Stochastic Processes

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Stochastic processes are probabilistic models of data streams such as speech, audio and video signals, stock market prices, and measurements of physical phenomena by digital sensors such as medical instruments, GPS receivers, or seismographs. A solid understanding of the mathematical basis of these models is essential for understanding phenomena and processing information in many branches of science and engineering including physics, communications, signal processing, automation, and structural dynamics.

These course notes introduce the theory of discrete-time multivariate stochastic processes (i.e. sequences of random vectors) that is needed for estimation and prediction. Students are assumed to have knowledge of basic probability and of matrix algebra. The course starts with a succinct review of the theory of discrete and continuous random variables and random vectors. Bayesian estimation of linear functions of multivariate normal (Gaussian) random vectors is introduced. There follows a presentation of random sequences, including discussions of convergence, ergodicity, and power spectral density. State space models of linear discrete-time dynamic systems are introduced, and their response to transient and stationary random inputs is studied. The estimation problem for linear discrete-time systems with normal (i.e. Gaussian) signals is introduced and the Kalman filter algorithm is derived.

Additional course materials, including exercise problems and recorded lectures, are available at the author's home page `http://www.tut.fi/~piche/stochastic`

Contents

1 Random Variables and Vectors

1.1 Discrete random variable

1.1.1 Specification

A *discrete* random variable X is a rv whose ensemble $\mathscr{E}_X$ is a countable (that is, finite or countably infinite) subset of $\mathbb{R}$ that does not contain limit points. Typical ensembles are $\mathbb{N}_n = \{1,2,\ldots,n\}$, $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N} = \{1,2,\ldots\}$, the non-negative integers $\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0,1,2,\ldots\}$.

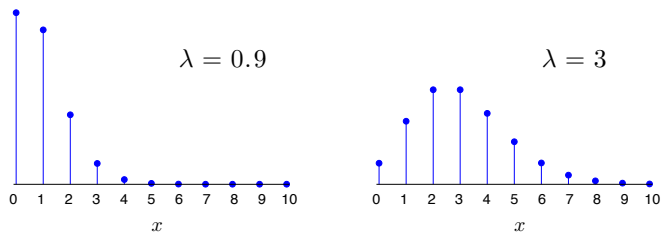
The probability law (or: probability distribution) of a discrete rv is specified by a *probability mass function* (pmf) $p_X(x) = \mathsf{P}(X = x)$, with $p_X(x) \geq 0$ and $\sum_{x \in \mathscr{E}_X} p_X(x) = 1$. (Recall that a sum's value is invariant to summation order iff the series converges absolutely.)

Degenerate distribution: If $\mathsf{P}(X = \mu) = 1$ for some μ then X is said to be equal to μ *almost surely*, and we write $X \overset{\text{as}}{=} \mu$.

Bernoulli distribution: If $\mathscr{E}_X = \mathbb{Z}_2 = \{0,1\}$ and $p_X(x) = \theta^x(1-\theta)^{1-x}$ then we write $X \sim \text{Bernoulli}(\theta)$. The parameter $\theta \in [0,1]$ is $\mathsf{P}(X = 1)$, the "chance of success".

Categorical distribution: If $\mathscr{E}_X = \mathbb{N}_n$ and $p_X(x) = \theta_x$ then we write $X \sim \text{Categorical}(\theta)$ or $X \sim \text{Categorical}(\theta_1, \ldots, \theta_n)$. The parameters $\theta_1, \theta_2, \ldots, \theta_n$ are nonnegative and sum to 1.

Poisson distribution: If $\mathscr{E}_X = \mathbb{Z}_+$ and $p_X(x) = \frac{1}{x!}\lambda^x \mathrm{e}^{-\lambda}$ then we write $X \sim \text{Poisson}(\lambda)$. $p_X(x)$ is the probability that x events occur in a unit time interval; the parameter $\lambda > 0$ is the "occurrence rate". Here are stem plots of two Poisson pmf's:



The cumulative distribution function (cdf) of a discrete rv is

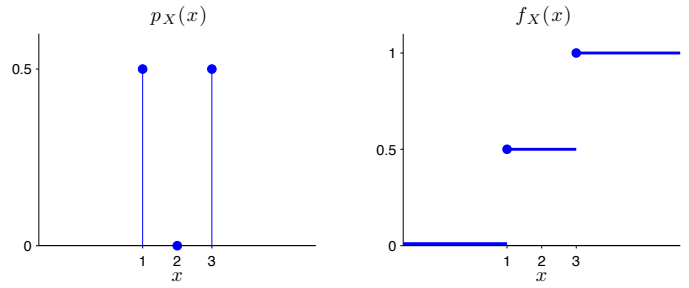
$$f_X(u) = \mathsf{P}(X \leq u) = \sum_{x \leq u} p_X(x)$$

Properties:

- f_X is piecewise constant, with jumps occurring at points of $\mathscr{E}_X$ that have positive pmf.
- $f_X(-\infty) = 0$
- $f_X(\infty) = 1$

- $\mathsf{P}(a < X \le b) = f_X(b) - f_X(a)$,
- $u \le v$ implies $f_X(u) \le f_X(v)$ (i.e. cdf's are nondecreasing),
- $\lim_{u \to a^+} f_X(u) = f_X(a)$ (i.e. cdf's are continuous from the right).

Example 1.1. Here are plots of the pmf and cdf of $X \sim \text{Categorical}(\frac{1}{2}, 0, \frac{1}{2})$:



■

Given a function $g : \mathscr{E}_X \to \mathbb{R}$, the transformed rv $Y = g(X)$ has ensemble $\mathscr{E}_Y \supseteq g(\mathscr{E}_X)$ and pmf

$$p_Y(y) = \sum_{\{x \in \mathscr{E}_X \,:\, g(x) = y\}} p_X(x).$$

In particular, if $Y = aX + b$ with $a \neq 0$ then $p_Y(y) = p_X(\frac{y-b}{a})$ when $\frac{y-b}{a} \in \mathscr{E}_X$, and $p_Y(y) = 0$ otherwise.

1.1.2 Expectation

If $\sum_{x \in \mathscr{E}_X} |x| p_X(x)$ converges, the *expected value* (or: mean, mean value, mathematical expectation, first moment) of a discrete rv X is $\mathsf{E}X = \sum_{x \in \mathscr{E}_X} x p_X(x)$, also denoted μ_X.

Example 1.2. For rv N with $\mathscr{E}_N = \mathbb{N}$ and $p_N(n) = \frac{1}{n(n+1)}$, the mean $\mathsf{E}N$ does not exist because the series $\sum_{n=1}^{\infty} n p_N(n)$ diverges. ■

Fundamental theorem of expectation: if $Y = g(X)$ and $\mathsf{E}Y$ exists then $\mathsf{E}Y = \sum_{x \in \mathscr{E}_X} g(x) p_X(x)$.

Example 1.3. Let $X \sim \text{Categorical}(\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{8})$ and let $Y = g(X)$ with

x	1	2	3	4	5
$g(x)$	-1	1	1	2	-1

Then $\mathscr{E}_Y \subseteq \{-1, 1, 2\}$ with $p_Y(-1) = \frac{1}{4}$, $p_Y(1) = \frac{3}{8}$, $p_Y(2) = \frac{3}{8}$, and

$$\mathsf{E}Y = -1 \cdot \frac{1}{4} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} = \frac{7}{8}$$

The fundamental theorem of expectation gives the same result, without the need to find the pmf of Y:

$$\mathsf{E}Y = g(1) \cdot \frac{1}{8} + g(2) \cdot \frac{1}{8} + g(3) \cdot \frac{1}{4} + g(4) \cdot \frac{3}{8} + g(5) \cdot \frac{1}{8} = \frac{7}{8}$$ ■

Expectation is linear: if $Y = g_1(X)$ and $Z = g_2(X)$ and $\mathsf{E}Y$ and $\mathsf{E}Z$ exist then

$$\mathsf{E}(\alpha Y + \beta Z) = \alpha(\mathsf{E}Y) + \beta(\mathsf{E}Z)$$

For any set $\mathscr{A} \subset \mathbb{R}$, $\mathsf{P}(X \in \mathscr{A}) = \mathsf{E}1_{\mathscr{A}}(X)$, where $1_{\mathscr{A}}$ denotes the *indicator function* of the set:

$$1_{\mathscr{A}}(a) = \begin{cases} 1 & \text{if } a \in \mathscr{A} \\ 0 & \text{otherwise} \end{cases}$$

The kth moment of X is $\mathsf{E}(X^k)$ (if the expectation exists). The root mean square (rms) value is denoted $\|X\|_{\text{rms}} := \sqrt{\mathsf{E}X^2}$.

If $\mathsf{E}(X^2)$ exists, then $\mathsf{E}X$ exists (because $|x| \le 1 + x^2$), and the *variance* of X is $\mathsf{var}X := \mathsf{E}(X - \mu_X)^2$, that is, the second moment of the centred rv $X - \mu_X$. The *standard deviation* of X is $\sigma_X := \sqrt{\mathsf{var}X}$.

Identities: $\mathsf{var}X = \mathsf{E}(X^2) - \mu_X^2 = \|X - \mu_X\|_{\text{rms}}^2$.

Monotonicity of expectation: If $\mathsf{P}(g(X) \le h(X)) = 1$ we say that $g(X) \le h(X)$ *almost surely* (or: with probability one). Then $\mathsf{E}g(X) \le \mathsf{E}h(X)$ (provided the expectations exist). In particular, if $X \ge 0$ almost surely then $\mathsf{E}X \ge 0$.

Expectation and variance of affine transformation: $\mathsf{E}(b + aX) = b + a\mathsf{E}X$ and $\mathsf{var}(b + aX) = a^2\,\mathsf{var}X$

The *mean square error* of $a \in \mathbb{R}$ with respect to a finite-variance rv X is $\text{mse}_X(a) := \|X - a\|_{\text{rms}}^2$.

Variational characterisation of the mean: μ_X is the unique minimizer of mse_X.

Proof. For any $a \in \mathbb{R}$ we have

$$\begin{aligned} \text{mse}_X(a) - \text{mse}_X(\mu_X) &= \|X - a\|_{\text{rms}}^2 - \|X - \mu_X\|_{\text{rms}}^2 \\ &= \mathsf{E}(X - a)^2 - \mathsf{E}(X - \mu_X)^2 = (a - \mu_X)^2 \ge 0 \end{aligned}$$

with equality only if $a \ne \mu_X$. ■

1.1.3 Inequalities

Markov's inequality: if $\mathsf{E}X$ exists and $\varepsilon > 0$ then $\mathsf{P}(|X| \ge \varepsilon) \le \frac{\mathsf{E}|X|}{\varepsilon}$.

Proof: let $\mathscr{A} = \{x : |x| \ge \varepsilon\}$. Then $|x| \ge |x| \cdot 1_{\mathscr{A}}(x) \ge \varepsilon \cdot 1_{\mathscr{A}}(x)$, and so $\mathsf{E}|X| \ge \mathsf{E}(\varepsilon\, 1_{\mathscr{A}}(X)) = \varepsilon\,\mathsf{E}(1_{\mathscr{A}}(X)) = \varepsilon\,\mathsf{P}(|X| \ge \varepsilon)$. ■

Example 1.4. At most 1% of the population can possess more than 100 times the average wealth, because $\mathsf{P}(|X| \ge 100\,\mathsf{E}|X|) \le \frac{1}{100}$. ■

Chebyshev's inequality: if $\mathsf{E}(X^2)$ exists and $\varepsilon > 0$ then $\mathsf{P}(|X| \ge \varepsilon) \le \varepsilon^{-2}\|X\|_{\text{rms}}^2$.

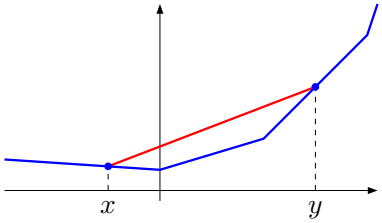
Corollary: if $0 < \alpha < 1$ then $\alpha \leq \mathsf{P}(|X - \mathsf{E}X| < \sqrt{\frac{\mathsf{var}X}{1-\alpha}})$, that is, the open interval $(\mathsf{E}X - \sqrt{\frac{\mathsf{var}X}{1-\alpha}},\ \mathsf{E}X + \sqrt{\frac{\mathsf{var}X}{1-\alpha}})$ contains at least α of the probability.

It follows that if $\mathsf{var}X = 0$ then $\mathsf{P}(X = \mathsf{E}X) = 1$ (denoted $X \overset{\text{as}}{=} \mathsf{E}X$).

A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$$

for all $x, y \in \mathbb{R}^n$ and all $\lambda \in [0,1]$.



Jensen's inequality: if $g : \mathbb{R} \to \mathbb{R}$ is convex and X has finite expectation then $g(\mathsf{E}X) \leq \mathsf{E}g(X)$.

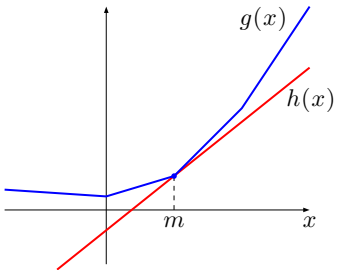
Proof:

By mathematical induction, the inequality in the definition of convexity can be extended to convex combinations of k points: if g is convex and $\theta_1, \ldots, \theta_k$ are non-negative and sum to 1, then

$$g(\theta_1 x_1 + \cdots + \theta_k x_k) \leq \theta_1 g(x_1) + \cdots \theta_k g(x_k)$$

for any $x_1, \ldots, x_k$. If X is a finite discrete rv with pmf $p_X(x_{1:k}) = \theta_{1:k}$, then the above inequality can be written as $g(\mathsf{E}X) \leq \mathsf{E}g(X)$. This inequality also holds for any rv for which X and $g(X)$ have finite expectation.

Jensen's inequality can also be derived using the fact that for any convex function g and any point m, there exists a *supporting line* $h(x) = g(m) + b \cdot (x - m)$ such that $g(x) \geq h(x)$ for all x.



In particular, using the supporting line at $m = \mathsf{E}X$, we have

$$\mathsf{E}g(X) \geq \mathsf{E}h(X) = \mathsf{E}(g(\mathsf{E}X) + b \cdot (X - \mathsf{E}X)) = g(\mathsf{E}X) \quad \blacksquare$$

Because the functions $x \mapsto |x|$ and $x \mapsto x^2$ are convex, then by Jensen's inequality we have $|\mathsf{E}X| \leq \mathsf{E}|X|$ and $(\mathsf{E}X)^2 \leq \mathsf{E}(X^2)$. The latter inequality can also be derived using the fact that

$$0 \leq \mathsf{var}(X) = \mathsf{E}(X^2) - (\mathsf{E}X)^2$$

1.1.4 Characteristic function

The *characteristic function* of X is

$$\phi_X(\zeta) = \mathsf{E}(\mathrm{e}^{\imath\zeta X}) = \mathsf{E}\cos(\zeta X) + \imath\mathsf{E}\sin(\zeta X) \qquad (\zeta \in \mathbb{R})$$

where $\imath$ denotes the imaginary unit, $\imath^2 = -1$.

Properties: $\phi_X(-\zeta) = \overline{\phi_X(\zeta)}$ and $|\phi_X(\zeta)| \leq \phi_X(0) = 1$.

> *Proof.* $0 \leq \mathsf{var}\cos(\zeta X) = \mathsf{E}\cos^2(\zeta X) - (\mathsf{E}\cos(\zeta X))^2$ and similarly for sin, so $1 = \mathsf{E}(\cos^2(\zeta X) + \sin^2(\zeta X)) \geq (\mathsf{E}\cos(\zeta X))^2 + (\mathsf{E}\sin(\zeta X))^2 = |\phi_X(\zeta)|^2$. ■

The kth moment of X is related to the kth derivative of ϕ_X by

$$\phi_X^{(k)}(0) = \imath^k \mathsf{E}X^k$$

If $X \sim \text{Bernoulli}(\theta)$, then $\phi_X(\zeta) = 1 - \theta + \theta\mathrm{e}^{\imath\zeta}$, $\mathsf{E}X = \theta$, $\mathsf{var}X = \theta(1-\theta)$

If $X \sim \text{Poisson}(\lambda)$, then $\phi_X(\zeta) = \mathrm{e}^{\lambda(\mathrm{e}^{\imath\zeta}-1)}$, $\mathsf{E}X = \lambda$, $\mathsf{var}X = \lambda$

1.2 Two discrete random variables

1.2.1 Specification

A *discrete bivariate* random vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has as ensemble the cartesian product of two countable point sets, $\mathscr{E}_X = \mathscr{E}_{X_1} \times \mathscr{E}_{X_2}$. Its probability distribution is specified by a (joint) pmf $p_X(x) = \mathsf{P}(X = x) = \mathsf{P}(X_1 = x_1, X_2 = x_2)$, with $p_X(x) \geq 0$ and $\sum_{x \in \mathscr{E}_X} p_X(x) = 1$.

Marginal rv: X_1 is a rv with ensemble $\mathscr{E}_{X_1}$ and pmf $p_{X_1}(x_1) = \sum_{x_2 \in \mathscr{E}_{X_2}} p_X(x)$, and similarly for X_2.

The rv's X_1 and X_2 are (statistically) *independent* if $p_X(x) = p_{X_1}(x_1)p_{X_2}(x_2)$ for all $x \in \mathscr{E}_X$.

The (joint) cdf is $f_X(u) = \mathsf{P}(X \leq u) = \sum_{x \leq u} p_X(x)$. Here $\leq$ is applied elementwise, i.e. $x \leq u$ means $x_1 \leq u_1$ and $x_2 \leq u_2$. f_X is piecewise constant with rectangular pieces, with jumps occurring on lines where $\mathscr{E}_{X_1}$ or $\mathscr{E}_{X_2}$ have positive pmf. Also,

$$f_X(\begin{bmatrix} u_1 \\ -\infty \end{bmatrix}) = 0, \quad f_X(\begin{bmatrix} -\infty \\ u_2 \end{bmatrix}) = 0, \quad f_X(\begin{bmatrix} \infty \\ \infty \end{bmatrix}) = 1,$$

$$f_{X_1}(u_1) = f_X(\begin{bmatrix} u_1 \\ \infty \end{bmatrix}), \quad f_{X_2}(u_2) = f_X(\begin{bmatrix} \infty \\ u_2 \end{bmatrix}),$$

$$\mathsf{P}(a < X \leq b) = f_X(b) - f_X(\begin{bmatrix} a_1 \\ b_2 \end{bmatrix}) - f_X(\begin{bmatrix} b_1 \\ a_2 \end{bmatrix}) + f_X(a),$$

$u \le v$ implies $f_X(u) \le f_X(v)$ (i.e. cdf's are nondecreasing), and $\lim_{u \to a^+} f_X(u) = f_X(a)$ (i.e. cdf's are continuous from the right). If X_1, X_2 are independent then $f_X(u) = f_{X_1}(u_1) f_{X_2}(u_2)$.

The transformed rv $Y = g(X)$, where $g : \mathscr{E}_X \to \mathbb{R}^m$, has ensemble $\mathscr{E}_Y \supseteq g_1(\mathscr{E}_X) \times \cdots \times g_m(\mathscr{E}_X)$ and pmf

$$p_Y(y) = \sum_{\{x \in \mathscr{E}_X \,:\, g(x) = y\}} p_X(x)$$

In particular, if $\mathbf{a}$ is a nonsingular 2×2 matrix and $Y = \mathbf{a}X + b$, then $p_Y(y) = p_X(\mathbf{a}^{-1}(y-b))$ when $\mathbf{a}^{-1}(y-b) \in \mathscr{E}_X$, otherwise $p_Y(y) = 0$.

Example 1.5. Let X_1 and X_2 be independent identically distributed (iid) rv's with ensemble $\mathbb{Z}$, and $Y = \max(X_1, X_2)$. Then

$$\begin{aligned} p_Y(y) &= p_{X_1}(y) p_{X_2}(y) + \sum_{x_1 < y} p_{X_1}(x_1) p_{X_2}(y) + \sum_{x_2 < y} p_{X_1}(y) p_{X_2}(x_2) \\ &= (p_{X_1}(y))^2 + 2 p_{X_1}(y) f_{X_1}(y-1) \end{aligned}$$ ■

1.2.2 Conditional distribution

Conditional rv: if $\begin{bmatrix} Y \\ Z \end{bmatrix}$ is a bivariate rv and $p_Z(z) > 0$, then $Y \,|\, (Z = z)$ is a scalar rv with ensemble $\mathscr{E}_Y$ and pmf

$$p_{Y|Z}(y|z) = \frac{p_{\begin{bmatrix} Y \\ Z \end{bmatrix}}(\begin{bmatrix} y \\ z \end{bmatrix})}{p_Z(z)}$$

If $g : \mathscr{E}_Z \to \mathscr{E}_Z$ is invertible then

$$p_{Y|g(Z)}(y|w) = p_{Y|Z}(y|g^{-1}(w))$$

If Y and Z are independent then $p_{Y|Z}(y|z) = p_Y(y)$ and $p_{Z|Y}(z|y) = p_Z(z)$.

Total probability theorem: $p_Y(y) = \sum_{z \in \mathscr{E}_Z} p_{Y|Z}(y|z) p_Z(z)$

Bayes' law (theorem):

$$p_{Y|Z}(y|z) = \frac{p_{Z|Y}(z|y) p_Y(y)}{p_Z(z)} = \frac{p_{Z|Y}(z|y) p_Y(y)}{\sum_{y \in \mathscr{E}_Y} p_{Z|Y}(z|y) p_Y(y)}$$

In Bayesian statistics, p_Y is the *prior* (distribution), $p_{Z|Y}$ is the *likelihood* (distribution), and $p_{Y|Z}$ is the *posterior* (distribution). In a statistical inference problem, the likelihood is a model of how an observable quantity Z depends on an unknown

parameter Y; the prior and posterior describe your state of knowledge about Y before and after you observe Z. Then Bayes' law is often written as

$$p_{Y|Z}(y|z) \propto p_{Z|Y}(z|y)p_Y(y)$$

with the proportionality constant is understood as being the normalisation factor that makes the posterior a proper probability distribution over $\mathscr{E}_Y$.

Example 1.6. Let Y be a sent bit and let Z be the received (decoded) bit. The transmission channel is characterised by the numbers $\lambda_{0|0}$, $\lambda_{0|1}$, $\lambda_{1|0}$, $\lambda_{1|1}$, which denote

$$\lambda_{z|y} := p_{Z|Y}(z|y) = \mathsf{P}(Z = z|Y = y) \qquad (y,z \in \{0,1\})$$

The two parameters $\lambda_{1|0}$ and $\lambda_{0|1}$, i.e. the probabilities of *false detection* and *missed detection*, suffice to characterise the transmission channel, because $\lambda_{0|0} + \lambda_{1|0} = 1$ and $\lambda_{0|1} + \lambda_{1|1} = 1$.

Suppose Y is a Bernoulli rv with parameter θ_1, and let $\theta_0 = 1 - \theta_1$. Then the posterior probability of $Y = y$, given an observation $Z = z$, is

$$\theta_{y|z} := p_{Y|Z}(y|z) = \frac{p_{Z|Y}(z|y)p_Y(y)}{\sum_{y\in\{0,1\}} p_{Z|Y}(z|y)p_Y(y)} = \frac{\lambda_{z|y}\theta_y}{\lambda_{z|0}\theta_0 + \lambda_{z|1}\theta_1}. \qquad \blacksquare$$

If Y and Z are independent then

$$p_{Y+Z}(s) = \sum_{z\in\mathscr{E}_Z} p_Y(s-z)p_Z(z)$$

which is a *convolution* when the points of $\mathscr{E}_Z$ are equally spaced.

If Y and Z are independent then so are $g_1(Y)$ and $g_2(Z)$ for any $g_1 : \mathscr{E}_Y \to \mathbb{R}$ and $g_2 : \mathscr{E}_Z \to \mathbb{R}$.

1.2.3 Expectation

The expected value of a bivariate discrete rv X is the two-element vector

$$\mathsf{E}X = \sum_{x\in\mathscr{E}_X} xp_X(x) = \begin{bmatrix} \mathsf{E}X_1 \\ \mathsf{E}X_2 \end{bmatrix}$$

and is also denoted μ_X.

Fundamental theorem of expectation: $\mathsf{E}g(X) = \sum_{x\in\mathscr{E}_X} g(x)p_X(x)$.

For any set $\mathscr{A} \subset \mathbb{R}^2$, $\mathsf{P}(X \in \mathscr{A}) = \mathsf{E}1_{\mathscr{A}}(X)$.

Expectation is linear: if $Y = g_1(X)$ and $Z = g_2(X)$ have the same dimensions and $\mathsf{E}Y$ and $\mathsf{E}Z$ exist then

$$\mathsf{E}(\alpha Y + \beta Z) = \alpha(\mathsf{E}Y) + \beta(\mathsf{E}Z)$$

The *correlation matrix*[1] of X is the 2×2 matrix

$$\mathsf{E}XX' = \begin{bmatrix} \mathsf{E}X_1^2 & \mathsf{E}X_1X_2 \\ \mathsf{E}X_2X_1 & \mathsf{E}X_2^2 \end{bmatrix}$$

where the prime denotes transpose. The correlation matrix is symmetric nonnegative definite.

The root-mean-square value of X is $\|X\|_{\text{rms}} = \sqrt{\mathsf{E}X'X}$. Note that $\|X\|_{\text{rms}}^2 = \operatorname{tr} \mathsf{E}XX' = \|X_1\|_{\text{rms}}^2 + \|X_2\|_{\text{rms}}^2$.

The rv's X_1 and X_2 are *orthogonal* if $\mathsf{E}X_1X_2 = 0$. If X_1, X_2 are orthogonal then $\|X_1 + X_2\|_{\text{rms}}^2 = \|X_1\|_{\text{rms}}^2 + \|X_2\|_{\text{rms}}^2$ (Pythagorean identity).

The *covariance matrix* of X is the 2×2 matrix

$$\mathsf{var}X = \mathsf{E}(X - \mu_X)(X - \mu_X)' = \begin{bmatrix} \mathsf{var}X_1 & \mathsf{cov}(X_1, X_2) \\ \mathsf{cov}(X_2, X_1) & \mathsf{var}X_2 \end{bmatrix}$$

where $\mathsf{cov}(X_1, X_2) = \mathsf{E}((X_1 - \mu_{X_1})(X_2 - \mu_{X_2}))$. The covariance matrix is symmetric nonnegative definite.

Identities: $\mathsf{var}X = \mathsf{E}XX' - \mu_X \mu_X'$, $\mathsf{cov}(X_1, X_2) = \mathsf{E}X_1X_2 - \mu_{X_1}\mu_{X_2}$, and $\|X - \mu_X\|_{\text{rms}}^2 = \operatorname{tr} \mathsf{var}X = \mathsf{var}X_1 + \mathsf{var}X_2$.

The rv's X_1 and X_2 are *uncorrelated* if $\mathsf{cov}(X_1, X_2) = 0$. If X_1, X_2 are independent then they are uncorrelated.

Example 1.7. This example illustrates that a pair of rv's may be uncorrelated but not independent. Consider the bivariate rv X having the pmf

$p_X(x)$	$x_2 = -1$	$x_2 = 0$	$x_2 = 1$
$x_1 = -1$	0	0.25	0
$x_1 = 0$	0.25	0	0.25
$x_1 = 1$	0	0.25	0

Then $\mathsf{cov}(X_1, X_2) = 0$ but $p_{X_1}(0) p_{X_2}(0) = \frac{1}{4} \neq p_X([0,0]')$. ■

Affine transformation: if $\mathbf{a}$ is an $m \times 2$ matrix and $b \in \mathbb{R}^m$ (with $m \in \{1, 2\}$) then $\mathsf{E}(b + \mathbf{a}X) = b + \mathbf{a}\,\mathsf{E}X$ and $\mathsf{var}(b + \mathbf{a}X) = \mathbf{a}\,(\mathsf{var}X)\,\mathbf{a}'$.

Consequently, $\mathsf{var}(X_1 + X_2) = \mathsf{var}([1,\, 1]X) = \mathsf{var}X_1 + \mathsf{var}X_2 + 2\mathsf{cov}(X_1, X_2)$. In particular, if X_1, X_2 are uncorrelated then $\mathsf{var}(X_1 + X_2) = \mathsf{var}X_1 + \mathsf{var}X_2$ (Bienaymé's identity).

Example 1.8. If X_1, X_2 are iid Bernoulli rv's then $\mathsf{E}(X_1 + X_2) = 2\theta$ and $\mathsf{var}(X_1 + X_2) = 2\theta(1 - \theta)$. ■

[1] Some books use this term to mean the matrix of correlation coefficients $\rho_{ij} = \frac{\mathsf{cov}(X_i, X_j)}{\sigma_{X_i} \sigma_{X_j}}$.

Principal components (or: discrete Karhunen-Loève transform): Let $\mathsf{var}X = \mathbf{u}\,\mathbf{d}\,\mathbf{u}'$ with $\mathbf{d}$ a nonnegative diagonal matrix and $\mathbf{u}$ an orthogonal matrix (eigenvalue factorisation). Then $Y = \mathbf{u}'(X - \mu_X)$ has $\mathsf{E}Y = 0$, $\mathsf{cov}(Y_1, Y_2) = 0$, and $\|Y\|_{\mathrm{rms}} = \|X - \mu_X\|_{\mathrm{rms}}$.

The *mean square error* of $a \in \mathbb{R}^2$ with respect to a rv X that has finite $\mathsf{var}X_1$ and $\mathsf{var}X_2$ is $\mathrm{mse}_X(a) := \|X - a\|^2_{\mathrm{rms}}$.

Variational characterisation of the mean: μ_X is the unique minimizer of mse_X.

Proof. For any $a \in \mathbb{R}^2$,

$$\begin{aligned}\mathrm{mse}_X(a) - \mathrm{mse}_X(\mathsf{E}X) &= \|X - a\|^2_{\mathrm{rms}} - \|X - \mu_X\|^2_{\mathrm{rms}} \\ &= \mathsf{E}(X-a)'(X-a) - \mathsf{E}(X-\mu_X)'(X-\mu_X) \\ &= \|a - \mu_X\|^2 \geq 0\end{aligned}$$

with equality only if $a = \mu_X$.

1.2.4 Conditional expectation

Let $\begin{bmatrix} Y \\ Z \end{bmatrix}$ be a bivariate discrete rv. If $p_Z(z) > 0$, the conditional expected value of Y given $Z = z$ is

$$\mathsf{E}(Y \,|\, Z = z) = \sum_{y \in \mathscr{E}_Y} y p_{Y|Z}(y|z)$$

This is a function $\mathscr{E}_Z \to \mathbb{R}$. The transformation of rv Z by this function is denoted $\mathsf{E}(Y \,|\, Z)$. The *law of total expectation* (or: law of iterated expectations) is

$$\mathsf{E}Y = \mathsf{E}(\mathsf{E}(Y \,|\, Z))$$

where the outer E is a sum over $\mathscr{E}_Z$ while the inner E is a sum over $\mathscr{E}_Y$.

The transformation of Z by the function $z \mapsto \mathsf{var}(Y \,|\, Z = z)$ is denoted $\mathsf{var}(Y \,|\, Z)$, and the *law of total variation* is

$$\mathsf{var}Y = \mathsf{var}(\mathsf{E}(Y \,|\, Z)) + \mathsf{E}(\mathsf{var}(Y \,|\, Z))$$

1.2.5 Inequalities

Cauchy-Schwartz inequality: $|\mathsf{E}X_1X_2| \leq \|X_1\|_{\mathrm{rms}} \cdot \|X_2\|_{\mathrm{rms}}$

Proof. $(\mathsf{E}X_1^2)(\mathsf{E}X_2^2) - (\mathsf{E}X_1X_2)^2 = \det(\mathsf{E}XX') \geq 0.$ ■

Triangle inequality: $\|X_1 + X_2\|_{\mathrm{rms}} \leq \|X_1\|_{\mathrm{rms}} + \|X_2\|_{\mathrm{rms}}$

Proof. $\|X_1 + X_2\|^2_{\mathrm{rms}} = \mathsf{E}(X_1 + X_2)^2 = \|X_1\|^2_{\mathrm{rms}} + 2\mathsf{E}X_1X_2 + \|X_2\|^2_{\mathrm{rms}} \leq \|X_1\|^2_{\mathrm{rms}} + 2|\mathsf{E}X_1X_2| + \|X_2\|^2_{\mathrm{rms}} \leq \|X_1\|^2_{\mathrm{rms}} + 2\|X_1\|_{\mathrm{rms}} \cdot \|X_2\|_{\mathrm{rms}} + \|X_2\|^2_{\mathrm{rms}} = (\|X_1\|_{\mathrm{rms}} + \|X_2\|_{\mathrm{rms}})^2.$ ■

Thus $\langle g_1, g_2 \rangle = \mathsf{E} g_1(X) g_2(X)$ is an inner product and $\|g(X)\|_{\text{rms}}$ is a norm on the vector space of finite-variance rv's $\mathscr{E}_X \to \mathbb{R}$.

Chebyshev inequality: if $\|X\|_{\text{rms}}$ exists and $\varepsilon > 0$ then $\mathsf{P}(\|X\| \geq \varepsilon) \leq \frac{\|X\|_{\text{rms}}^2}{\varepsilon^2}$.

Corollary: if $0 < \alpha < 1$ then $\mathsf{P}(\|X - \mathsf{E}X\|^2 < \frac{\text{tr}\,\mathsf{var}X}{1-\alpha}) \geq \alpha$, that is, the open disk with centre at $\mathsf{E}X$ and radius $\sqrt{\frac{\text{tr}\,\mathsf{var}X}{1-\alpha}}$ contains at least α of the probability.

If $\mathsf{var}X$ is singular (that is, if $\det \mathsf{var}X = \mathsf{var}X_1 \cdot \mathsf{var}X_2 - (\mathsf{cov}(X_1, X_2))^2 = 0$) then all the probability lies on a line in the plane (or, if $\mathsf{var}X = 0$, at a point).

Proof. The set of a such that $(\mathsf{var}X)\,a = 0$ is denoted $\text{null}(\mathsf{var}X)$. If $\mathsf{var}X$ is singular, $\text{null}(\mathsf{var}X)$ is a line through the origin or, if $\mathsf{var}X = 0$, the entire plane. If $a \in \text{null}(\mathsf{var}X)$ then $\|a'(X - \mathsf{E}X)\|_{\text{rms}}^2 = \mathsf{var}(a'X) = a'(\mathsf{var}X)a = 0$, and by Chebyshev's inequality $\mathsf{P}(\|a'(X - \mathsf{E}X)\| \geq \varepsilon) = 0$ for any $\varepsilon > 0$, and so $a'(X - \mu_X) \overset{\text{as}}{=} 0$. Thus, the probability mass of X lies in a line through μ_X that is perpendicular to the nullspace line (or, if $\mathsf{var}X = 0$, entirely at μ_X). ■

Jensen's inequality: if $g : \mathbb{R}^2 \to \mathbb{R}$ is convex and X has finite expectation then $g(\mathsf{E}X) \leq \mathsf{E}g(X)$.

Consequently, $\|\mathsf{E}X\| \leq \mathsf{E}\|X\|$.

1.2.6 Characteristic function

The (joint) characteristic function of a bivariate rv X is

$$\phi_X(\zeta) = \mathsf{E}\exp(\imath \zeta' X) = \mathsf{E}\exp(\imath \zeta_1 X_1 + \imath \zeta_2 X_2) \quad (\zeta \in \mathbb{R}^2)$$

Marginal cf:

$$\phi_{X_1}(\zeta_1) = \phi_X(\begin{bmatrix} \zeta_1 \\ 0 \end{bmatrix})$$

The moments of X are related to the partial derivatives $\phi_X^{(k_1,k_2)} = \frac{\partial^{k_1+k_2}\phi_X}{\partial \zeta_1^{k_1} \partial \zeta_2^{k_2}}$ by the formula

$$\phi_X^{(k_1,k_2)}(0) = \imath^{k_1+k_2} \mathsf{E} X_1^{k_1} X_2^{k_2}$$

If X_1 and X_2 are independent then the cf of the sum is

$$\phi_{X_1+X_2}(\zeta) = \phi_{X_1}(\zeta)\phi_{X_2}(\zeta)$$

Example 1.9. The sum of two independent Poisson rv's is a Poisson rv with rate parameter $\lambda_1 + \lambda_2$, because

$$\phi_{X_1+X_2}(\zeta) = \mathrm{e}^{\lambda_1(\mathrm{e}^{\imath\zeta}-1)} \mathrm{e}^{\lambda_2(\mathrm{e}^{\imath\zeta}-1)} = \mathrm{e}^{(\lambda_1+\lambda_2)(\mathrm{e}^{\imath\zeta}-1)} \qquad ■$$

1.3 Discrete random vector

1.3.1 Specification

A discrete *multivariate random variable* (random vector) is a rv X whose ensemble is the cartesian product of n countable point sets,

$$\mathscr{E}_X = \mathscr{E}_{X_1} \times \cdots \times \mathscr{E}_{X_n} = \prod_{i=1}^{n} \mathscr{E}_{X_i}$$

Its probability distribution is specified by a (joint) pmf $p_X(x) = \mathsf{P}(X = x) = \mathsf{P}(X_1 = x_1, \ldots, X_n = x_n)$, with $p_X(x) \geq 0$ and $\sum_{x \in \mathscr{E}_X} p_X(x) = 1$.

Marginal rv: X_1 is the scalar rv with ensemble $\mathscr{E}_{X_1}$ and pmf

$$p_{X_1}(x_1) = \sum_{x_2 \in \mathscr{E}_{X_2}} \cdots \sum_{x_n \in \mathscr{E}_{X_n}} p(x)$$

Similarly, the pmf X_i is obtained by "summing out" the $n-1$ other dimensions.

More generally, $X_{1:k}$ is the k-variate rv with ensemble $\prod_{i=1}^{k} \mathscr{E}_{X_i}$ and pmf

$$p_{X_{1:k}}(x_{1:k}) = \sum_{x_{k+1} \in \mathscr{E}_{X_{k+1}}} \cdots \sum_{x_n \in \mathscr{E}_{X_n}} p_X(x)$$

The rv's $X_1, \ldots, X_n$ are mutually independent if $p_X(x) = p_{X_1}(x_1) p_{X_2}(x_2) \cdots p_{X_n}(x_n)$ for all $x \in \mathscr{E}_X$.

The m rv's $X_{1:k_1}, X_{k_1+1:k_2}, \ldots, X_{k_{m-1}+1:n}$ are mutually independent if

$$p_X(x) = p_{X_{1:k_1}}(x_{1:k_1}) p_{X_{k_1+1:k_2}}(x_{k_1+1:k_2}) \cdots p_{X_{k_{m-1}+1:n}}(x_{k_{m-1}+1:n})$$

for all $x \in \mathscr{E}_X$.

The (joint) cdf is $f_X(u) = \mathsf{P}(X \leq u) = \sum_{x \leq u} p_X(x)$, where $\leq$ is applied elementwise. f_X is piecewise constant with "hyper-rectangular" pieces, with jumps occurring on hyperplanes where the marginal ensembles $\mathscr{E}_{X_i}$ have positive pmf. Also,

$$\lim_{u_k \to -\infty} f_X(u) = 0, \qquad \lim_{\substack{u_1 \to \infty \\ \vdots \\ u_n \to \infty}} f_X(u) = 1, \quad f_{X_{1:k}}(u_{1:k}) = f_X\left(\begin{bmatrix} u_{1:k} \\ \infty \\ \vdots \\ \infty \end{bmatrix} \right),$$

$u \leq v$ implies $f_X(u) \leq f_X(v)$ (i.e. cdf's are nondecreasing), and $\lim_{u \to a^+} f_X(u) = f_X(a)$ (i.e. cdf's are right-continuous). If $X_{1:k_1}, X_{k_1+1:k_2} \ldots, X_{k_{m-1}+1:n}$ are mutually independent then

$$f_X(u) = f_{X_{1:k_1}}(u_{1:k_1}) f_{X_{k_1+1:k_2}}(u_{k_1+1:k_2}) \cdots f_{X_{k_{m-1}+1:n}}(u_{k_{m-1}+1:n})$$

The transformed rv $Y = g(X)$, where $g : \mathscr{E}_X \to \mathbb{R}^m$, has ensemble $\mathscr{E}_Y \supseteq g_1(\mathscr{E}_X) \times \cdots \times g_m(\mathscr{E}_X)$ and pmf

$$p_Y(y) = \sum_{\{x \in \mathscr{E}_X \,:\, g(x)=y\}} p_X(x)$$

In particular, if $\mathbf{a}$ is a nonsingular $n \times n$ matrix and $Y = \mathbf{a}X + b$, then $p_Y(y) = p_X(\mathbf{a}^{-1}(y-b))$ when $\mathbf{a}^{-1}(y-b) \in \mathscr{E}_X$, otherwise $p_Y(y) = 0$.

1.3.2 Conditional distribution

Conditional rv: if $\begin{bmatrix} Y \\ Z \end{bmatrix}$ is a partition of a multivariate rv and $p_Z(z) > 0$, then $Y \,|\, (Z = z)$ is an rv with ensemble $\mathscr{E}_Y$ and pmf

$$p_{Y|Z}(y|z) = \frac{p_{\begin{bmatrix} Y \\ Z \end{bmatrix}}(\begin{bmatrix} y \\ z \end{bmatrix})}{p_Z(z)}$$

If Y and Z are independent then $p_{Y|Z}(y|z) = p_Y(y)$ and $p_{Z|Y}(z|y) = p_Z(z)$.

Total probability theorem:

$$p_Y(y) = \sum_{z \in \mathscr{E}_Z} p_{Y|Z}(y|z) p_Z(z)$$

Bayes' theorem (or: Bayes' law) is

$$p_{Y|Z}(y|z) = \frac{p_{Z|Y}(z|y) p_Y(y)}{p_Z(z)} = \frac{p_{Z|Y}(z|y) p_Y(y)}{\sum_{y \in \mathscr{E}_Y} p_{Z|Y}(z|y) p_Y(y)}$$

Example 1.6 (continued from §1.2.2). Suppose the bit Y is transmitted twice, and that the received bits Z_1, Z_2 are conditionally independent given $Y = y$, that is,

$$p_{Z_{1:2}|Y}(z_{1:2}|y) = p_{Z_1|Y}(z_1|y) p_{Z_2|Y}(z_2|y)$$

Assume also that the transmission channel model $\lambda_{z_i|y}$ is the same for both received bits. Then the posterior probability pmf of Y, given the observations $Z_1 = z_1, Z_2 = z_2$, can be computed recursively, one observation at a time:

$$\theta_{y|z_1} = p_{Y|Z_1}(y|z_1) \propto p_{Z_1|Y}(z_1|y) p_Y(y) = \lambda_{z_1|y} \theta_y$$
$$\theta_{y|z_{1:2}} = p_{Y|Z_{1:2}}(y|z_{1:2}) \propto p_{Z_2|Y}(z_2|y) p_{Y|Z_1}(y|z_1) = \lambda_{z_2|y} \theta_{y|z_1} \quad \blacksquare$$

If Y and Z are independent and of the same size then

$$p_{Y+Z}(s) = \sum_{z \in \mathscr{E}_Z} p_Y(s-z) p_Z(z)$$

This sum is called a *convolution* when the points of $\mathscr{E}_Z$ are equally spaced.

If Y and Z are independent then so are $g_1(Y)$ and $g_2(Z)$ for any $g_1 : \mathscr{E}_Y \to \mathbb{R}^{n_1}$ and $g_2 : \mathscr{E}_Z \to \mathbb{R}^{n_2}$. More generally, if $X_{1:k_1}, X_{k_1+1:k_2}, \ldots, X_{k_{m-1}+1:n}$ are mutually independent then so are $g_1(X_{1:k_1}), g_2(X_{k_1+1:k_2}), \ldots, g_m(X_{k_{m-1}+1:n})$.

1.3.3 Expectation

The expected value of a discrete multivariate rv X is the n-element vector

$$\mathsf{E}X = \sum_{x \in \mathscr{E}_X} x p_X(x) = \begin{bmatrix} \mathsf{E}X_1 \\ \vdots \\ \mathsf{E}X_n \end{bmatrix}$$

Fundamental theorem of expectation: $\mathsf{E}g(X) = \sum_{x \in \mathscr{E}_X} g(x) p_X(x)$.

For any set $\mathscr{A} \subset \mathbb{R}^n$, $\mathsf{P}(X \in \mathscr{A}) = \mathsf{E}1_{\mathscr{A}}(X)$

Expectation is linear: if $Y = g_1(X)$ and $Z = g_2(X)$ have the same dimensions and $\mathsf{E}Y$ and $\mathsf{E}Z$ exist then

$$\mathsf{E}(\alpha Y + \beta Z) = \alpha(\mathsf{E}Y) + \beta(\mathsf{E}Z)$$

The *correlation matrix* of X is the $n \times n$ matrix

$$\mathsf{E}XX' = \begin{bmatrix} \mathsf{E}X_1^2 & \mathsf{E}X_1X_2 & \cdots & \mathsf{E}X_1X_n \\ \mathsf{E}X_2X_1 & \mathsf{E}X_2^2 & \cdots & \mathsf{E}X_2X_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}X_nX_1 & \mathsf{E}X_nX_2 & \cdots & \mathsf{E}X_n^2 \end{bmatrix}$$

The correlation matrix is symmetric nonnegative definite.

We denote $\|X\|_{\text{rms}}^2 = \mathsf{E}X'X$. Then $\|X\|_{\text{rms}}^2 = \operatorname{tr}\mathsf{E}XX' = \|X_1\|_{\text{rms}}^2 + \cdots + \|X_n\|_{\text{rms}}^2$.

The rv's $Y = g_1(X)$ and $Z = g_2(X)$ are *orthogonal* if they are the same size and $\mathsf{E}Y'Z = 0$. If Y, Z are orthogonal then $\|Y+Z\|_{\text{rms}}^2 = \|Y\|_{\text{rms}}^2 + \|Z\|_{\text{rms}}^2$ (Pythagorean identity).

The *covariance matrix* of X is the $n \times n$ matrix

$$\begin{aligned} \mathsf{var}X &= \mathsf{E}(X-\mu_X)(X-\mu_X)' \\ &= \begin{bmatrix} \mathsf{var}X_1 & \mathsf{cov}(X_1,X_2) & \cdots & \mathsf{cov}(X_1,X_n) \\ \mathsf{cov}(X_2,X_1) & \mathsf{var}X_2 & \cdots & \mathsf{cov}(X_2,X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{cov}(X_n,X_1) & \mathsf{cov}(X_n,X_2) & \cdots & \mathsf{var}X_n \end{bmatrix} \end{aligned}$$

where $\mathrm{cov}(X_i, X_j) = \mathsf{E}(X_i - \mathsf{E}X_i)(X_j - \mathsf{E}X_j)$. The covariance matrix is symmetric nonnegative definite.

The *cross covariance* of $Y = g_1(X)$ and $Z = g_2(X)$ is the $n_Y \times n_Z$ matrix $\mathrm{cov}(Y,Z) = \mathsf{E}((Y-\mu_Y)(Z-\mu_Z)')$. The rv's Y and Z are *uncorrelated* if $\mathrm{cov}(Y,Z) = 0$. If Y, Z are independent then they are uncorrelated.

Identities: $\mathrm{var}X = \mathsf{E}XX' - \mu_X\mu_X'$, $\mathrm{cov}(Y,Z) = (\mathrm{cov}(Z,Y))' = \mathsf{E}YZ' - \mu_Y\mu_Z'$, and $\|X - \mu_X\|_{\mathrm{rms}}^2 = \mathrm{tr}\,\mathrm{var}X = \mathrm{var}X_1 + \cdots + \mathrm{var}X_n$.

Affine transformation: if $\mathbf{a}$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$ then $\mathsf{E}(b + \mathbf{a}X) = b + \mathbf{a}\mathsf{E}X$ and $\mathrm{var}(b + \mathbf{a}X) = \mathbf{a}(\mathrm{var}X)\mathbf{a}'$.

Consequently,

$$\mathrm{var}(X_1 + \cdots + X_n) = \mathrm{var}([1, \cdots, 1]X) = \sum_{i=1}^{n} \left(\mathrm{var}X_i + 2 \sum_{j=i+1}^{n} \mathrm{cov}(X_i, X_j) \right)$$

In particular, if $X_1, \cdots, X_n$ are uncorrelated (i.e. $\mathrm{var}X$ is a diagonal matrix) then $\mathrm{var}(X_1 + \cdots + X_n) = \mathrm{var}X_1 + \cdots \mathrm{var}X_n$ (Bienaymé's identity). Similarly, if Y and Z are the same size and are uncorrelated then $\mathrm{var}(Y+Z) = \mathrm{var}(Y) + \mathrm{var}(Z)$.

> *Example 1.10.* If $X_1, \ldots, X_n$ are iid Bernoulli rv's then $\mathsf{E}(X_1 + \cdots + X_n) = n\theta$ and $\mathrm{var}(X_1 + \cdots + X_n) = n\theta(1-\theta)$. ■

Principal components, discrete Karhunen-Loève transform: Let $\mathrm{var}X = \mathbf{u}\mathbf{d}\mathbf{u}'$ be an eigenvalue factorisation. Then $Y = \mathbf{u}'(X - \mu_X)$ has $\mathsf{E}Y = 0$, $Y_1, \ldots, Y_n$ are uncorrelated, and $\|Y\|_{\mathrm{rms}} = \|X - \mu_X\|_{\mathrm{rms}}$.

The *mean square error* of $a \in \mathbb{R}^n$ with respect to a rv X that has finite $\mathrm{var}X_1, \ldots, \mathrm{var}X_n$ is $\mathrm{mse}_X(a) := \|X - a\|_{\mathrm{rms}}^2$.

Variational characterisation of the mean: μ_X is the unique minimizer of mse_X.

1.3.4 Conditional expectation

The statements of §1.2.4 hold with $\begin{bmatrix} Y \\ Z \end{bmatrix}$ denoting a multivariate discrete rv.

1.3.5 Inequalities

Cauchy-Schwartz inequality and Triangle inequality: if $Y = g(X)$ and $Z = h(X)$ have the same dimensions then $|\mathsf{E}Y'Z| \le \|Y\|_{\mathrm{rms}} \cdot \|Z\|_{\mathrm{rms}}$ and $\|Y+Z\|_{\mathrm{rms}} \le \|Y\|_{\mathrm{rms}} + \|Z\|_{\mathrm{rms}}$.

> *Proof:* $(\mathsf{E}Y'Y)(\mathsf{E}Z'Z) - (\mathsf{E}Y'Z)^2 = \det \mathsf{E} \begin{bmatrix} Y' \\ Z' \end{bmatrix} \begin{bmatrix} Y & Z \end{bmatrix} \ge 0$
> and $\|Y+Z\|_{\mathrm{rms}}^2 = \mathsf{E}(Y+Z)'(Y+Z) = \|Y\|_{\mathrm{rms}}^2 + 2\mathsf{E}Y'Z + \|Z\|_{\mathrm{rms}}^2 \le \|Y\|_{\mathrm{rms}}^2 + 2\|Y\|_{\mathrm{rms}}\|Z\|_{\mathrm{rms}} + \|Z\|_{\mathrm{rms}}^2 = (\|Y\|_{\mathrm{rms}} + \|Z\|_{\mathrm{rms}})^2$. ■

Consequently, $\langle g_1, g_2 \rangle = \mathsf{E} g_1(X)' g_2(X)$ is an inner product and $\|g(X)\|_{\text{rms}}$ is a norm on the vector space of finite-variance functions of X.

Chebyshev inequality: if $\|X\|_{\text{rms}}$ exists and $\varepsilon > 0$ then $\mathsf{P}(\|X\| \geq \varepsilon) \leq \frac{\|X\|_{\text{rms}}^2}{\varepsilon^2}$.

Corollary: if $0 < \alpha < 1$ then $\mathsf{P}(\|X - \mathsf{E}X\|^2 < \frac{\text{tr}\,\mathsf{var}X}{1-\alpha}) \geq \alpha$, that is, the open ball with centre at $\mathsf{E}X$ and radius $\sqrt{\frac{\text{tr}\,\mathsf{var}X}{1-\alpha}}$ contains at least α of the probability.

If $\mathsf{var}X$ is singular then $a'(X - \mu_X) \stackrel{\text{as}}{=} 0$ for $a \in \text{null}(\mathsf{var}X)$, that is, all the probability lies in a proper affine subspace of $\mathbb{R}^n$.

Jensen's inequality: if $g : \mathbb{R}^n \to \mathbb{R}$ is convex and X has finite expectation then $g(\mathsf{E}X) \leq \mathsf{E} g(X)$.

Consequently, $\|\mathsf{E}X\| \leq \mathsf{E}\|X\|$.

1.3.6 Characteristic function

The (joint) characteristic function of a multivariate rv X is

$$\phi_X(\zeta) = \mathsf{E} \exp(\imath \zeta' X) \quad (\zeta \in \mathbb{R}^n)$$

The marginal cf of X_i is $\phi_{X_i}(\zeta_i) = \phi_X(\zeta_i e[i])$ where $e[i]$ is the ith column of the $n \times n$ identity matrix. The marginal cf of $X_{1:k}$ is

$$\phi_{X_{1:k}}(\zeta_{1:k}) = \phi_X\left(\begin{bmatrix} \zeta_{1:k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

The moments of X are related to the partial derivatives $\phi_X^{(k_1,\ldots,k_n)} = \frac{\partial^{k_1+\cdots+k_n} \phi_X}{\partial \zeta_1^{k_1} \cdots \partial \zeta_n^{k_n}}$ by the formula

$$\phi_X^{(k_1,\ldots,k_n)}(0) = \imath^{k_1+\cdots+k_n} \mathsf{E} X_1^{k_1} \cdots X_n^{k_n}$$

If $Y = g(X)$ and $Z = h(X)$ are of the same dimension and independent then the cf of the sum is

$$\phi_{Y+Z}(\zeta) = \phi_Y(\zeta) \phi_Z(\zeta)$$

Example 1.11. The sum $Y = X_1 + \cdots + X_n$ of n iid Bernoulli rv's is

$$\phi_Y(\zeta) = (1 - \theta + \theta \mathrm{e}^{\imath\zeta})^n = \sum_{y=0}^{n} \binom{n}{y} (1-\theta)^{n-y} \theta^y \mathrm{e}^{\imath y \zeta}$$

which is the cf of a rv with ensemble $\mathbb{Z}_{n+1}$ and pmf

$$p_Y(y) = \binom{n}{y} (1-\theta)^{n-y} \theta^y$$

This is called a *binomial* rv, and we write $Y \sim \text{Binomial}(\theta, n)$.

If $Y_1 \sim \text{Binomial}(\theta, n_1)$ and $Y_2 \sim \text{Binomial}(\theta, n_2)$ are independent, then $Y_1 + Y_2 \sim \text{Binomial}(\theta, n_1 + n_2)$, because

$$\phi_{Y_1+Y_2}(\zeta) = (1-\theta+\theta e^{\imath\zeta})^{n_1}(1-\theta+\theta e^{\imath\zeta})^{n_2} = (1-\theta+\theta e^{\imath\zeta})^{n_1+n_2} \qquad \blacksquare$$

1.4 Continuous random variable

1.4.1 Specification

An *(absolutely) continuous* random variable X is a rv with ensemble $\mathscr{E}_X = \mathbb{R}$ whose probability law is specified by a *probability density function* (pdf) $p_X : \mathbb{R} \to [0,\infty)$ such that $\int_{\mathbb{R}} p_X(x)\,dx = 1$. For any Borel set $\mathscr{A} \subset \mathbb{R}$, $\mathsf{P}(X \in \mathscr{A}) = \int_{\mathscr{A}} p_X(x)\,dx$. (Countable unions and intersections of intervals are Borel sets.)

Uniform distribution: If $\mathscr{A} \subset \mathbb{R}$ is a Borel set with finite length $\int_{\mathscr{A}} dx =: |\mathscr{A}| > 0$ and $p_X(x) = \frac{1_{\mathscr{A}}(x)}{|\mathscr{A}|}$, then we write $X \sim \text{Uniform}(\mathscr{A})$. The *standard uniform* rv has $\mathscr{A} = [0,1]$.

Normal (or Gaussian) distribution: If

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

then we write $X \sim \text{Normal}(\mu, \sigma^2)$. The *standard normal* rv has $\mu = 0$ and $\sigma^2 = 1$.

The cumulative distribution function (cdf) of a continuous rv is $f_X(u) = \mathsf{P}(X \le u) = \int_{x \le u} p_X(x)\,dx$. f_X is continuous, and $p_X(x) = f_X'(x)$ at points where p_X is continuous. Also, $f_X(-\infty) = 0$, $f_X(\infty) = 1$, $\mathsf{P}(a < X \le b) = f_X(b) - f_X(a)$, and $u \le v$ implies $f_X(u) \le f_X(v)$ (i.e. cdf's are nondecreasing).

The cdf of a standard normal rv is denoted $\Phi(u)$. It is related to the *error function*

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-x^2}\,dx$$

by the identity $\Phi(u) = \frac{1}{2} + \frac{1}{2}\text{erf}(\frac{u}{\sqrt{2}})$. Also, $\Phi(-u) = 1 - \Phi(u)$ because the pdf is an even function.

If $g : \mathbb{R} \to \mathbb{R}$ is differentiable with $g' < 0$ everywhere or $g' > 0$ everywhere then $Y = g(X)$ is a continuous rv with pdf

$$p_Y(y) = \frac{1}{|g'(g^{-1}(y))|} p_X(g^{-1}(y))$$

This rule can be informally written as $p_Y|\,dy| = p_X|\,dx|$, "the probability of an event is preserved under a change of variable".

In particular, if $Y = aX + b$ with $a \neq 0$ then $p_Y(y) = \frac{1}{|a|} p_X(\frac{y-b}{a})$. Thus if $X \sim \text{Uniform}([0,1])$ and $a > 0$ then $aX + b \sim \text{Uniform}([b, b+a])$.

1.4.2 Expectation, inequalities, characteristic function

The results of sections 1.1.2–4 hold, with summation replaced by integration. Note that the characteristic function ϕ_X is the inverse Fourier transform of the pdf p_X.

If $X \sim \text{Uniform}([0,1])$ then $\phi_X(\zeta) = \frac{e^{\imath\zeta}-1}{\imath\zeta}$, $\mathsf{E}X = \frac{1}{2}$, $\mathsf{var}X = \frac{1}{12}$

If $X \sim \text{Normal}(\mu, \sigma^2)$ then $\phi_X(\zeta) = e^{\imath\mu\zeta - \frac{1}{2}\sigma^2\zeta^2}$, $\mathsf{E}X = \mu$, $\mathsf{var}X = \sigma^2$, and $aX + b \sim \text{Normal}(b + a\mu, a^2\sigma^2)$.

1.4.3 Median

A *median* of a continuous or discrete rv $X$is a real number m such that at least half of the probability lies above m and at least half of the probability lies below m, that is, $\mathsf{P}(X \le m) \ge \frac{1}{2}$ and $\mathsf{P}(m \le X) \ge \frac{1}{2}$. Equivalently, m is a median of X if $\mathsf{P}(X < m) \le \frac{1}{2}$ and $\mathsf{P}(m < X) \le \frac{1}{2}$.

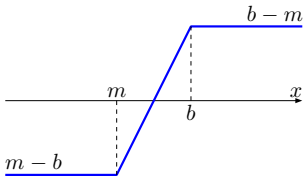
Any rv has at least one median. If f_X is continuous then the median is unique and is $m = f_X^{-1}(\frac{1}{2})$, the 50% quantile. Otherwise, a rv may have more than one median. For example, any of the values $\{1,2,3\}$ is a median of $X \sim \text{Categorical}(\frac{1}{2}, 0, \frac{1}{2})$.

The *mean absolute error* of a with respect to X is $\text{mae}_X(a) := \mathsf{E}|X-a|$.

Variational characterisation of the median: for any finite-expectation rv X, m is a median of X iff m is a minimizer of mae_X.

Proof. For any m and for any $b > m$ we have

$$|X-m| - |X-b| = \begin{cases} m-b & \text{if } X \le m \\ 2X-(m+b) & \text{if } m < X < b \\ b-m & \text{if } b \le X \end{cases}$$



and so

$$\left\{ \begin{array}{ll} m-b & \text{if } X < b \\ b-m & \text{if } b \le X \end{array} \right\} \le |X-m| - |X-b| \le \left\{ \begin{array}{ll} m-b & \text{if } X \le m \\ b-m & \text{if } m < X \end{array} \right\}$$

Then

$$\begin{aligned} \text{mae}_X(m) - \text{mae}_X(b) &= \mathsf{E}(|X-m| - |X-b|) \\ &\le \mathsf{E}\big((m-b)1_{(-\infty,m]}(X) + (b-m)1_{(m,\infty)}(X)\big) \\ &= (m-b)\mathsf{P}(X \le m) + (b-m)\mathsf{P}(m < X) \\ &= (b-m)(2\mathsf{P}(m < X) - 1) \end{aligned}$$

and

$$\begin{aligned}\mathrm{mae}_X(m) - \mathrm{mae}_X(b) &= \mathsf{E}(|X-m| - |X-b|) \\ &\geq \mathsf{E}\big((m-b)1_{(-\infty,b)}(X) + (b-m)1_{[b,\infty)}(X)\big) \\ &= (m-b)\mathsf{P}(X<b) + (b-m)\mathsf{P}(b\leq X) \\ &= (b-m)(2\mathsf{P}(b\leq X)-1)\end{aligned}$$

For any $a < m$ we can similarly show that

$$(m-a)(2\mathsf{P}(X\leq a)-1) \leq \mathrm{mae}_X(m) - \mathrm{mae}_X(a) \leq (m-a)(2\mathsf{P}(X<m)-1)$$

Now, if m is a median of X, we have $\mathsf{P}(X<m)\leq\frac{1}{2}$ and $\mathsf{P}(m<X)\leq\frac{1}{2}$, which, using the above inequalities, implies $\mathrm{mae}_X(m)\leq\mathrm{mae}_X(a)$ and $\mathrm{mae}_X(m)\leq\mathrm{mae}_X(b)$. Thus, m minimizes mae_X.

Conversely, if m minimizes mae_X, we have $0\leq\mathrm{mae}_X(m)-\mathrm{mae}_X(a)$ and $0\leq\mathrm{mae}_X(m)-\mathrm{mae}_X(b)$, which, using the above results, implies $\mathsf{P}(X\leq a)\leq\frac{1}{2}$ and $\mathsf{P}(b\leq X)\leq\frac{1}{2}$. Because these inequalities hold for any $a<m<b$, we have $\mathsf{P}(X<m)\leq\frac{1}{2}$ and $\mathsf{P}(m<X)\leq\frac{1}{2}$, and so m is a median of X. ■

Any median m of X is within one standard deviation of the mean of X, that is, $|m-\mu_X|\leq\sqrt{\mathsf{var}X}$.

Proof.[2] Using Jensen's inequality and the fact that m minimizes mae_X, we have

$$|m-\mu_X| = |\mathsf{E}(X-m)| \leq \mathsf{E}|X-m| \leq \mathsf{E}|X-\mu_X|$$

Using Jensen's inequality again, we have

$$(\mathsf{E}|X-\mu_X|)^2 \leq \mathsf{E}(|X-\mu_X|^2) = \mathsf{E}((X-\mu_X)^2) = \mathsf{var}X \quad ■$$

1.5 Continuous random vector

1.5.1 Specification

An *(absolutely) continuous* multivariate rv (random vector) X has ensemble $\mathscr{E}_X = \mathbb{R}^n$ and probability law specified by a pdf $p_X:\mathbb{R}^n\to[0,\infty)$ such that $\int_{\mathbb{R}^n} p_X(x)\,\mathrm{d}x = 1$. For any Borel set $\mathscr{A}\subset\mathbb{R}^n$, $\mathsf{P}(X\in\mathscr{A}) = \int_{\mathscr{A}} p_X(x)\,\mathrm{d}x$.

Marginal rv's are obtained by "integrating out" the remaining dimensions.

Mutual independence is defined as for discrete rv's, with pdf's in place of pmf's.

[2] C. Mallows, *The American Statistician*, August 1991, Vol. 45, No. 3, 1991, p. 257.

The (joint) cdf is $f_X(u) = \mathsf{P}(X \le u) = \int_{x \le u} p_X(x)\,\mathrm{d}x$, where $\le$ is applied elementwise. f_X is continuous, and

$$p_X(x) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} f_X(x)$$

at points where p_X is continuous. Also,

$$\lim_{u_k \to -\infty} f_X(u) = 0, \qquad \lim_{\substack{u_1 \to \infty \\ \vdots \\ u_n \to \infty}} f_X(u) = 1, \quad f_{X_{1:k}}(u_{1:k}) = f_X(\begin{bmatrix} u_{1:k} \\ \infty \\ \vdots \\ \infty \end{bmatrix}),$$

and $u \le v$ implies $f_X(u) \le f_X(v)$ (i.e. cdf's are nondecreasing). If $X_{1:k_1}, X_{k_1+1:k_2} \ldots, X_{k_{m-1}+1:n}$ are mutually independent then

$$f_X(u) = f_{X_{1:k_1}}(u_{1:k_1}) f_{X_{k_1+1:k_2}}(u_{k_1+1:k_2}) \cdots f_{X_{k_{m-1}+1:n}}(u_{k_{m-1}+1:n})$$

If $g : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and differentiable then $Y = g(X)$ is a continuous rv with pdf

$$p_Y(y) = \frac{1}{|\det \mathbf{j}(g^{-1}(y))|} p_X(g^{-1}(y))$$

where

$$\mathbf{j} = \begin{bmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial g_n/\partial x_1 & \cdots & \partial g_n/\partial x_n \end{bmatrix}$$

In particular, if $Y = \mathbf{a}X + b$ with nonsingular $\mathbf{a} \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ then $p_Y(y) = \frac{1}{|\det \mathbf{a}|} p_X(\mathbf{a}^{-1}(y-b))$.

1.5.2 Conditional distribution, expectation, inequalities, characteristic function

The results of sections 1.2.2–6 and 1.3.2–6 hold, with summation replaced by integration.

1.5.3 Multivariate normal distribution

Consider n iid standard normal rv's iid $U_1, \ldots, U_n$. The cf of the multivariate rv U is

$$\phi_U(\omega) = \phi_{U_1}(\omega_1) \cdots \phi_{U_n}(\omega_n) = \mathrm{e}^{-\frac{1}{2}\omega_1^2} \cdots \mathrm{e}^{-\frac{1}{2}\omega_n^2} = \mathrm{e}^{-\frac{1}{2}\omega'\omega}$$

An m-variate rv obtained by an affine mapping $X = b + \mathbf{a}U$ of iid standard normal rv's is called a multivariate normal distribution. Its cf is

$$\phi_X(\zeta) = \mathsf{E}\mathrm{e}^{\imath\zeta'(b+\mathbf{a}U)} = \mathrm{e}^{\imath\zeta' b}\mathsf{E}\mathrm{e}^{\imath(\mathbf{a}'\zeta)'U} = \mathrm{e}^{\imath\zeta' b}\phi_U(\mathbf{a}'\zeta) = \mathrm{e}^{\imath\zeta' b - \frac{1}{2}\zeta'\mathbf{a}\mathbf{a}'\zeta}$$

Its mean is $\mathsf{E}X = b$ and its variance is $\mathsf{var}X = \mathbf{a}\mathbf{a}'$, and we write $X \sim \text{Normal}(b, \mathbf{a}\mathbf{a}')$.

$\mathsf{var}X = \mathbf{a}\mathbf{a}'$ is nonsingular (positive definite) iff $\mathbf{a}$ has full rank m. If $\mathsf{var}X$ is nonsingular then X is a continuous rv and its pdf is

$$p_X(x) = \frac{1}{(2\pi)^{m/2}\sqrt{\det(\mathsf{var}X)}} \mathrm{e}^{-\frac{1}{2}(x-\mathsf{E}X)'(\mathsf{var}X)^{-1}(x-\mathsf{E}X)}$$

If $\mathsf{var}X$ is singular then X is not a continuous rv in $\mathbb{R}^m$, because all its probability is contained in a proper affine subspace of $\mathbb{R}^m$.

Example 1.12. If $U \sim \text{Normal}(0,1)$ and $X = \begin{bmatrix} 2U+1 \\ U \end{bmatrix}$ then $\mathsf{var}X = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, which has rank 1. The probability of X is contained in the line $x_1 = 2x_2 + 1$. ■

If $X \sim \text{Normal}(b, \mathbf{c})$ is an m-variate continuous rv, $\alpha \in (0,1)$, and $\text{chi2inv}(\cdot, m)$ denotes the inverse cumulative distribution function of the chi-squared distribution with m degrees of freedom, then the ellipsoid

$$\mathscr{E} = \left\{x : (x-b)'\mathbf{c}^{-1}(x-b) < \text{chi2inv}(\alpha, m)\right\}$$

contains α of the probability of X, that is, $\mathsf{P}(X \in \mathscr{E}) = \alpha$. In particular, because $\text{chi2inv}(0.95, 1) = 1.96^2$, an interval containing 95% of the probability of a univariate non-degenerate normal $X \sim \text{Normal}(b, \sigma^2)$ is

$$\mathscr{E} = \left\{x : \frac{(x-b)^2}{\sigma^2} < 1.96^2\right\} = b \pm 1.96\sigma$$

Also, because $\text{chi2inv}(0.95, 2) = 5.99$, 95% of the probability of a bivariate continuous normal $X \sim \text{Normal}(b, \mathbf{c})$ is contained in the ellipse

$$\mathscr{E} = \left\{x : (x-b)'\mathbf{c}^{-1}(x-b) < 5.99\right\}$$

In MATLAB or Octave, this ellipse can be plotted with the code

```
[u,s,v]=svd(5.99*c); t=0:0.02:2*pi;
e=u*sqrt(s)*[cos(t);sin(t)];
plot(b(1)+e(1,:),b(2)+e(2,:)); axis equal
```

An affine transformation $Y = d + \mathbf{c}X$ of a normal rv X is normal with $\mathsf{E}Y = d + \mathbf{c}\,\mathsf{E}X$ and $\mathsf{var}Y = \mathbf{c}\,\mathsf{var}X\,\mathbf{c}'$.

The marginal rv $X_{1:k}$ of a normal rv X is a normal rv with $\mathsf{E}X_{1:k} = (\mathsf{E}X)_{1:k}$ and $\mathsf{var}X_{1:k} = (\mathsf{var}X)_{1:k,1:k}$.

If Y and Z are jointly normal and uncorrelated then Y, Z are independent. More generally, if $\mathsf{var}X$ is block diagonal then the corresponding m marginal rv's $X_{1:k_1}$, $X_{k_1+1:k_2}, \ldots, X_{k_{m-1}+1:m}$ are mutually independent.

Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \text{Normal}\left(\begin{bmatrix} m_x \\ m_y \end{bmatrix}, \begin{bmatrix} \mathbf{c}_x & \mathbf{c}_{xy} \\ \mathbf{c}_{yx} & \mathbf{c}_y \end{bmatrix} \right)$$

with nonsingular $\mathbf{c}_y$. Then

$$X \,|\, (Y = y) \sim \text{Normal}\left(m_x + \mathbf{c}_{xy}\mathbf{c}_y^{-1}(y - m_y),\ \mathbf{c}_x - \mathbf{c}_{xy}\mathbf{c}_y^{-1}\mathbf{c}_{yx}\right)$$

Proof. Let $Z = X - m_x - \mathbf{c}_{xy}\mathbf{c}_y^{-1}(Y - m_y)$. This rv is an affine combination of jointly normal rv's and so is normal. Its mean is $\mathsf{E}Z = \mathsf{E}\big(X - m_y - \mathbf{c}_{xy}\mathbf{c}_y^{-1}(Y - m_y)\big) = 0$ and its covariance is

$$\mathbf{c}_z = \begin{bmatrix} \mathbf{i} & -\mathbf{c}_{xy}\mathbf{c}_y^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c}_x & \mathbf{c}_{xy} \\ \mathbf{c}_{yx} & \mathbf{c}_y \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ -\mathbf{c}_y^{-1}\mathbf{c}_{yx} \end{bmatrix} = \mathbf{c}_x - \mathbf{c}_{xy}\mathbf{c}_y^{-1}\mathbf{c}_{yx}$$

The rv's Y and Z are independent, because

$$\mathsf{cov}(Y, Z) = \mathsf{E}(Y - m_y)(X - m_x - \mathbf{c}_{xy}\mathbf{c}_y^{-1}(Y - m_y))' = \mathbf{0}$$

and so

$$\begin{aligned} X \,|\, (Y = y) &= \big(Z + m_x + \mathbf{c}_{xy}\mathbf{c}_y^{-1}(Y - m_y)\big) \,|\, (Y = y) \\ &= m_x + \mathbf{c}_{xy}\mathbf{c}_y^{-1}(y - m_y) + \underbrace{Z \,|\, (Y = y)}_{\sim \text{Normal}(0, \mathbf{c}_z)} \end{aligned}$$

that is, the conditional rv is the sum of a constant and a zero-mean normal rv with covariance $\mathbf{c}_z$. ■

Example 1.12 (continued). For

$$X \sim \text{Normal}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \right)$$

the conditional rv $X_1 \,|\, (X_2 = x_2)$ is

$$X_1 \,|\, (X_2 = x_2) \sim \text{Normal}(2x_2 + 1, 0)$$

All its probability is concentrated on the point $2x_2 + 1$. Here, knowledge of the realized value of X_2 determines, with probability 1, the value of X_1. ■

1.5.4 Spatial Median

The *mean absolute error* of $a \in \mathbb{R}^n$ with respect to the n-variate continuous or discrete random vector X is $\text{mae}_X(a) := \mathsf{E}\|X - a\|$. A *spatial median* (or: L_1-median) of X is an n-vector m that is a minimizer of mae_X, that is, $\text{mae}_X(m) \leq \text{mae}_X(a)$ for any n-vector a.

If the probability of X is not entirely contained in a line in $\mathbb{R}^n$ then the spatial median of X is unique.

Proof.[3] Suppose there are two spatial medians $a \neq b$ of X, so that $\mathrm{mae}_X(a) = \mathrm{mae}_X(b) = \min \mathrm{mae}_X$, and let ℓ be the line passing through them. Then for every $x \in \mathbb{R}^n$ we have, by the triangle inequality,

$$\|x - \tfrac{1}{2}a - \tfrac{1}{2}b\| = \|\tfrac{1}{2}(x-a) + \tfrac{1}{2}(x-b)\| \leq \tfrac{1}{2}\|x-a\| + \tfrac{1}{2}\|x-b\|$$

Using the fact that the triangle inequality is an equality iff the vectors are parallel, we have

$$\|x - \tfrac{1}{2}a - \tfrac{1}{2}b\| < \tfrac{1}{2}\|x-a\| + \tfrac{1}{2}\|x-b\|$$

for every $x \in \mathbb{R}^n - \ell$. Then, because the probability of X is not contained in ℓ, the strict inequality is preserved when we take expectations, and we have

$$\mathsf{E}\|X - \tfrac{1}{2}a - \tfrac{1}{2}b\| < \tfrac{1}{2}\mathsf{E}\|X-a\| + \tfrac{1}{2}\mathsf{E}\|X-b\|$$

that is,

$$\mathrm{mae}_X(\tfrac{1}{2}a + \tfrac{1}{2}b) < \tfrac{1}{2}\mathrm{mae}_X(a) + \tfrac{1}{2}\mathrm{mae}_X(b) = \min \mathrm{mae}_X$$

which is a contradiction. ■

If m is a spatial median of X and $\mathsf{E}X$ exists then $\|\mathsf{E}X - m\| \leq \sqrt{\mathrm{tr}\,\mathsf{var}X}$.

Proof.

$$\begin{aligned}
\|\mathsf{E}X - m\| &= \|\mathsf{E}(X-m)\| \\
&\leq \mathsf{E}\|X-m\| && \text{(Jensen)} \\
&\leq \mathsf{E}\|X-\mathsf{E}X\| && \text{(spatial median definition)} \\
&\leq \sqrt{\mathsf{E}\|X-\mathsf{E}X\|^2} && \text{(Jensen)} \\
&= \sqrt{\mathsf{E}\,\mathrm{tr}(X-\mathsf{E}X)'(X-\mathsf{E}X)} \\
&= \sqrt{\mathrm{tr}\,\mathsf{var}X} \qquad \blacksquare
\end{aligned}$$

1.6 Estimation

Mathematical models are typically created to help understand real-world data. *Estimation* is the process of determining (inferring) the parameters of a mathematical model by comparing real-world observations with the corresponding results predicted by the model. Because of the various idealizations and approximations that are made during the modelling process, we don't expect real-world observations to agree exactly with the model. Indeed, observations differ even when the same experiments are repeated with all the conditions, as far as we can determine, identical. In statistical approaches to estimation, this variability is taken into account by modelling the observation as a random variable.

[3] P. Milasevic & G. R. Ducharme, Uniqueness of the spatial median, *The Annals of Statistics*, 1987, vol. 15, no. 3, pp. 1332–1333, `http://projecteuclid.org/euclid.aos/1176350511`

In Bayesian estimation, the quantities being estimated are also modelled as random variables. This approach is based on the idea that the laws of probability serve as a logically consistent way of modelling one's state of knowledge about these values.

This section introduces the basic theory of estimating parameters in a linear model when all rv's are normal (Gaussian). There is also a brief discussion of non-Gaussian estimation for data that contains large outliers.

1.6.1 Linear measurements and normal distributions

The linear measurement model is

$$Y = \mathbf{h}X + V$$

where Y represents observations (measurements), X represents parameters that we want to infer, V represents "noise", and $\mathbf{h}$ is a known matrix. We assume that X and V are uncorrelated jointly normal rv's, with $X \sim \text{Normal}(\hat{x}^-, \mathbf{p}^-)$ (the *prior* distribution) and $V \sim \text{Normal}(0, \mathbf{r})$ with nonsingular $\mathbf{r}$.

An alternative, equivalent description of the situation is to say that the measurements are conditionally independent samples of a normal distribution having mean $\mathbf{h}x$ and variance $\mathbf{r}$, that is,

$$Y \,|\, (X = x) \sim \text{Normal}(\mathbf{h}x, \mathbf{r})$$

Because the transformation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ \mathbf{h}X + V \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{h} & \mathbf{i} \end{bmatrix} \begin{bmatrix} X \\ V \end{bmatrix}$$

is linear, the joint distribution of (X, Y) is normal, with

$$\mathsf{E} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \hat{x}^- \\ \mathbf{h}\hat{x}^- \end{bmatrix}, \quad \mathsf{var} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{p}^- & \mathbf{p}^-\mathbf{h}' \\ \mathbf{h}\mathbf{p}^- & \mathbf{h}\mathbf{p}^-\mathbf{h}' + \mathbf{r} \end{bmatrix}$$

The solution of the estimation problem is a straightforward application of the formula for conditional normal rv's given in §1.5.3: our state of knowledge about the parameters, given the observed values, is completely described by the *posterior distribution* $X \,|\, (Y = y) \sim \text{Normal}(\hat{x}^+, \mathbf{p}^+)$ with

$$\begin{aligned} \hat{x}^+ &= \hat{x}^- + \mathbf{k}\left(y - \mathbf{h}\hat{x}^-\right) \\ \mathbf{p}^+ &= \mathbf{p}^- - \mathbf{k}\mathbf{h}\mathbf{p}^- \\ \mathbf{k} &= \mathbf{p}^-\mathbf{h}'\left(\mathbf{h}\mathbf{p}^-\mathbf{h}' + \mathbf{r}\right)^{-1} \qquad \text{"gain"} \end{aligned}$$

The posterior mean $\hat{x}^+$ is an "optimal estimate" in the sense that it minimizes $\text{mse}_{X \,|\, (Y=y)}$. $\hat{x}^+$ is also optimal in the sense that it is the mode of the posterior density, that is, it is the *maximum a-posteriori* (MAP) value.

Independent measurements can be processed one at a time, as follows. Let a sequence of measurements be modelled as

$$\begin{aligned} Y[1] &= \mathbf{h}[1]X + V[1] \\ &\vdots \\ Y[k] &= \mathbf{h}[k]X + V[k] \end{aligned}$$

with $X, V[1], \ldots, V[k]$ jointly normal and uncorrelated. The $V[j]$, which may be of different dimensions, are zero-mean with nonsingular covariances $\mathbf{r}[j]$; also $X \sim \text{Normal}(\hat{x}[0], \mathbf{p}[0])$. Then the corresponding posteriors are

$$X \,|\, (Y[1{:}j] = y[1{:}j]) \sim \text{Normal}(\hat{x}[j], \mathbf{p}[j]) \qquad (j \in \mathbb{N}_k)$$

with

$$\hat{x}[j] = \hat{x}[j-1] + \mathbf{k}[j]\big(y[j] - \mathbf{h}[j]\hat{x}[j-1]\big) \tag{1a}$$

$$\mathbf{p}[j] = \mathbf{p}[j-1] - \mathbf{k}[j]\,\mathbf{h}[j]\,\mathbf{p}[j-1] \tag{1b}$$

$$\mathbf{k}[j] = \mathbf{p}[j-1]\,\mathbf{h}[j]'\big(\mathbf{h}[j]\,\mathbf{p}[j-1]\,\mathbf{h}[j]' + \mathbf{r}[j]\big)^{-1} \tag{1c}$$

Note that the observed value $y[j]$ does not appear in the formulas for the gains and posterior covariances.

Example 1.13. Gollum comes to the edge of a deep chasm. He fearfully peers over the edge: the chasm appears to be 150 ± 50 m deep! He clicks his tongue and hears the echo from the bottom 0.88 ± 0.03 s later. He clicks again and this time hears the echo 0.85 ± 0.03 s later. How deep does he estimate the chasm to be?

Let's model Gollum's prior belief as $X \sim \text{Normal}(150, 50^2)$. Using elementary physics, the echo times are modelled as

$$Y[j] = \frac{2X}{c} + V[j] \qquad (j \in \{1,2\})$$

with sound speed $c = 343$ m/s and $V[1]$, $V[2]$ zero-mean normal rv's with variance $(.03)^2$, uncorrelated to each other and to X.

The first observation leads to a normal posterior with

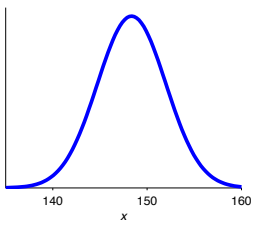
$$\hat{x}[1] = 150.91, \quad \mathbf{p}[1] = 26.194$$

The second observation updates the posterior parameters to

$$\hat{x}[2] = 148.36, \quad \mathbf{p}[2] = 13.166$$

Gollum's posterior distribution is therefore

$$X \,|\, (Y[1{:}2] = y[1{:}2]) \sim \text{Normal}(148.36, 13.166)$$



In particular, because

$$148.36 \pm 1.96\sqrt{13.166} \approx [141, 156],$$

Gollum now believes that it is 95% probable that the chasm's depth x is between 141 m and 156 m. In other words, Gollum believes that 19 to 1 are *fair odds* to offer on a bet against $x \in [141, 156]$. ■

1.6.2 Estimating a mean

A special case of the model presented in §1.6.1 is

$$Y[j] = X + V[j] \qquad (j \in \mathbb{N}_k)$$

with $V[j] \overset{\text{iid}}{\sim} \text{Normal}(0, \mathbf{r})$. Here X, $Y[j]$, and $V[j]$ are random n-vectors, $\mathbf{r}$ is still assumed to be nonsingular, and the $V[j]$ are assumed to be uncorrelated to X.

An alternative, equivalent description of the situation is to say that the measurements are conditionally independent samples of a normal distribution having mean x and variance $\mathbf{r}$, that is,

$$Y[j] \,|\, (X = x) \overset{\text{iid}}{\sim} \text{Normal}(x, \mathbf{r})$$

Given a prior distribution $X \sim \text{Normal}(\hat{x}[0], \mathbf{p}[0])$, the posterior distribution $X \,|\, (Y[1{:}k] = y[1{:}k]) \sim \text{Normal}(\hat{x}[k], \mathbf{p}[k])$ can be found by processing the measurements one at a time with the formulas

$$\begin{aligned} \hat{x}[j] &= \hat{x}[j-1] + \mathbf{p}[j-1]\big(\mathbf{p}[j-1] + \mathbf{r}\big)^{-1}(y[j] - \hat{x}[j-1]) \\ \mathbf{p}[j] &= \mathbf{p}[j-1] - \mathbf{p}[j-1]\big(\mathbf{p}[j-1] + \mathbf{r}\big)^{-1}\mathbf{p}[j-1] \\ &= \mathbf{p}[j-1]\big(\mathbf{p}[j-1] + \mathbf{r}\big)^{-1}\mathbf{r} \end{aligned}$$

for $j = 1, 2, \ldots, k$.

The posterior variance is bounded by

$$\operatorname{tr}\mathbf{p}[k] \leq \frac{\operatorname{tr}(\mathbf{r})}{k}$$

Thus, you can obtain 10 times better accuracy (i.e. 10 times smaller standard deviations) by using 100 times more samples.

Proof: As can be verified by mathematical induction, the posterior covariance matrix is given by

$$\mathbf{p}[k] = \mathbf{p}[0]\big(\mathbf{r} + k\mathbf{p}[0]\big)^{-1}\mathbf{r}$$

Let $\mathbf{u}$ be a nonsingular matrix and and $\mathbf{d}$ a diagonal non-negative definite matrix such that $\mathbf{u}'\mathbf{p}[0]\mathbf{u} = \mathbf{d}$ and $\mathbf{u}'\mathbf{r}\mathbf{u} = \mathbf{i}$ (generalized eigenvalue factorisation). Then

$$\mathbf{u}'\mathbf{p}[k]\mathbf{u} = \mathbf{u}'\mathbf{p}[0]\big(\mathbf{r} + k\mathbf{p}[0]\big)^{-1}\mathbf{r}\mathbf{u} = \mathbf{d}(\mathbf{i} + k\mathbf{d})^{-1}$$

is a diagonal matrix whose diagonal terms are all $\leq \frac{1}{k}$, so

$$\begin{aligned}\mathrm{tr}(\mathbf{p}[k]) &= \mathrm{tr}\big((\mathbf{u}')^{-1}\mathbf{d}(\mathbf{i}+k\mathbf{d})^{-1}\mathbf{u}^{-1}\big) = \mathrm{tr}\big(\mathbf{u}^{-1}(\mathbf{u}')^{-1}\mathbf{d}(\mathbf{i}+k\mathbf{d})^{-1}\big) \\ &\leq \frac{1}{k}\mathrm{tr}\big(\mathbf{u}^{-1}(\mathbf{u}')^{-1}\big) = \frac{1}{k}\mathrm{tr}\big((\mathbf{u}')^{-1}\mathbf{u}^{-1}\big) = \frac{1}{k}\mathrm{tr}(\mathbf{r}) \qquad \blacksquare\end{aligned}$$

1.6.3 Method of weighted least squares

If the prior covariance $\mathbf{p}^-$ in §1.6.1 is nonsingular, then, using the matrix inversion lemma (or: Sherman-Morrison-Woodbury formula)

$$(\mathbf{a}+\mathbf{bdc})^{-1} = \mathbf{a}^{-1} - \mathbf{a}^{-1}\mathbf{b}(\mathbf{d}^{-1}+\mathbf{ca}^{-1}\mathbf{b})^{-1}\mathbf{ca}^{-1}$$

the solution of the estimation problem in §1.6.1 can be written

$$\mathbf{p}^+ = \big((\mathbf{p}^-)^{-1} + \mathbf{h}'\mathbf{r}^{-1}\mathbf{h}\big)^{-1} \tag{2a}$$

$$\mathbf{k} = \mathbf{p}^+\mathbf{h}'\mathbf{r}^{-1} \tag{2b}$$

$$\hat{x}^+ = \mathbf{p}^+(\mathbf{p}^-)^{-1}\hat{x}^- + \mathbf{k}y \tag{2c}$$

These update formulas may be preferred for example in cases where $\mathbf{h}$ has many more columns than rows, because then the matrix to be inverted in computing $\mathbf{p}^+$ is smaller than in the formulas in §1.6.1.

If $\mathbf{p}^-$ is nonsingular and $\mathbf{h}$ is 1-1 (i.e. is injective, has null space $\{0\}$, has linearly independent columns, is left-invertible), then as $(\mathbf{p}^-)^{-1} \to \mathbf{0}$ the update formulas (2) become

$$\mathbf{p}^+ = \big(\mathbf{h}'\mathbf{r}^{-1}\mathbf{h}\big)^{-1} \tag{3a}$$

$$\mathbf{k} = \mathbf{p}^+\mathbf{h}'\mathbf{r}^{-1} \tag{3b}$$

$$\hat{x}^+ = \mathbf{k}y \tag{3c}$$

The prior Normal$(\hat{x}^-, \mathbf{p}^-)$ with $(\mathbf{p}^-)^{-1} \to \mathbf{0}$ corresponds to the improper "flat" prior $p_X(x) \propto 1$. The update formulas (3) coincide with the formulas of the *weighted least squares* method, and so we shall refer to the linear-normal estimation problem with the flat prior as a WLS problem.

To process a sequence of measurements in a WLS problem one at a time, (3) can be used to compute $\hat{x}[1]$ and $\mathbf{p}[1]$; then $\hat{x}[j]$, $\mathbf{p}[j]$ for $j = 2, 3, \ldots$ can be computed using either the update formulas (1) or the equivalent formulas (2).

In particular, when $\mathbf{h} = \mathbf{i}$ (as in §1.6.2), the WLS problem's solution is the arithmetic average (sample mean)

$$\hat{x}[k] = \frac{1}{k}\sum_{j=1}^{k} y[j]$$

Example 1.13 (continued): Treating Gollum's observations as a WLS problem, the first observation leads to

$$\hat{x}[1] = 150.92, \quad \mathbf{p}[1] = 26.471$$

The second observation updates the posterior parameters to

$$\hat{x}[2] = 148.35, \quad \mathbf{p}[2] = 13.236$$

We see that in this case, the solution of WLS problem (that is, the estimation problem with the "improper" prior $(\mathbf{p}[0])^{-1} \to \mathbf{0}$) is quite close to the solution of the estimation problem whose prior was proper with large variance. ■

Example 1.14.[4] Consider the problem of estimating the amplitude and phase of a sinusoidal signal of known frequency from a sequence of noisy samples,

$$Y_i = F(t_i) + V_i \quad (i \in \{1,2,\ldots,n\})$$

The n sampling instants are equally-spaced in the interval $t \in [-\frac{1}{2}l, \frac{1}{2}l]$ with sampling period $\delta = \frac{l}{n-1}$ (with $\delta < \frac{\pi}{\omega}$), that is,

$$t_i = \frac{(i-1)l}{n-1} - \frac{l}{2} \quad (i \in \{1,2,\ldots,n\})$$

The signal is

$$F(t) = X_1 \cos(\omega t) + X_2 \sin(\omega t)$$

Assuming that the samples are conditionally independent given X, and that the noise is $V \sim \text{Normal}(\mathbf{0}, r\mathbf{i})$, the observation model is

$$Y \,|\, (X = x) \sim \text{Normal}(\mathbf{h}x, r\mathbf{i})$$

where

$$\mathbf{h} = \begin{bmatrix} \cos(\omega \mathbf{t}) & \sin(\omega \mathbf{t}) \end{bmatrix} = \begin{bmatrix} \cos(\omega t_1) & \sin(\omega t_1) \\ \cos(\omega t_2) & \sin(\omega t_2) \\ \vdots & \vdots \\ \cos(\omega t_n) & \sin(\omega t_n) \end{bmatrix}$$

Assuming a "flat" prior and applying the summation formulas

$$\sum_{k=0}^{n-1} \cos(\phi + k\alpha) = \frac{\sin(n\alpha/2)\cos(\phi + \frac{1}{2}(n-1)\alpha)}{\sin(\alpha/2)}$$
$$\sum_{k=0}^{n-1} \sin(\phi + k\alpha) = \frac{\sin(n\alpha/2)\sin(\phi + \frac{1}{2}(n-1)\alpha)}{\sin(\alpha/2)}$$

yields the posterior distribution $X \,|\, (Y = y) \sim \text{Normal}(\hat{x}^+, \mathbf{p}^+)$ with

$$\mathbf{p}^+ = r \begin{bmatrix} \frac{n}{2} + \frac{\sin(n\omega\delta)}{2\sin(\omega\delta)} & 0 \\ 0 & \frac{n}{2} - \frac{\sin(n\omega\delta)}{2\sin(\omega\delta)} \end{bmatrix}^{-1}, \quad \hat{x}^+ = \frac{1}{r}\mathbf{p}^+ \begin{bmatrix} \sum_{i=1}^n \cos(\omega t_i) y_i \\ \sum_{i=1}^n \sin(\omega t_i) y_i \end{bmatrix}$$

[4] G. L. Bretthort, *Bayesian Spectrum Analysis and Parameter Estimation*, 1988, `http://bayes.wustl.edu/glb/book.pdf`

Note that (thanks to our clever choice of time-origin) the posterior distributions of X_1 and X_2 are uncorrelated.

Because the term $\frac{\sin(n\omega\delta)}{2\sin(\omega\delta)}$ appearing in $\mathbf{p}^+$ becomes negligible compared to $\frac{n}{2}$ for large n, we can use the approximation

$$\mathbf{p}^+ \approx \frac{2r}{n}\mathbf{i}, \quad \hat{x}^+ \approx \frac{2}{n}\begin{bmatrix} \sum_{i=1}^n y_i \cos(\omega t_i) \\ \sum_{i=1}^n y_i \sin(\omega t_i) \end{bmatrix}$$

The approximate formulas for $\hat{x}^+$ are known as *periodogram* formulas, and can be efficiently computed using FFT. ■

1.6.4 Robust estimation

Consider, as in §1.6.2, a measurement system modelled as

$$Y[j] = X + V[j] \qquad (j = 1, 2, \ldots, k)$$

Instead of assuming normal-distributed noise, we now assume the noises $V[j]$ to be iid with the pdf

$$p_{V[j]}(v) \propto \mathrm{e}^{-\beta\|v\|} \qquad (\beta > 0)$$

This is a multivariate generalisation of the Laplace (or: double exponential) distribution. Assuming the measurements to be conditionally independent given X, we have

$$p_{Y[1:k]\,|\,X}(y[1{:}k]\,|\,x) \propto \mathrm{e}^{-\beta\sum_{j=1}^k \|y[j]-x\|}$$

Assuming a "flat" prior $p_X(x) \propto 1$, we obtain from Bayes' theorem the posterior distribution

$$p_{X\,|\,Y}(x\,|\,y) \propto \mathrm{e}^{-\beta\sum_{j=1}^k \|y[j]-x\|}$$

The maximum a-posteriori (MAP) estimate is

$$\hat{x} = \arg\min_x \sum_{j=1}^k \|y[j] - x\|$$

The MAP estimate is thus the *sample spatial median* of the data, that is, the spatial median of the discrete probability distribution having the cdf

$$f(u) = \frac{\text{number of observations } y[\cdot] \text{ that are } \leq u}{k}$$

We have shown that when the noise model is changed from normal to Laplace, the MAP estimate (with flat prior) changes from sample mean to sample spatial median. An advantage of the sample spatial median over the sample mean is that it less sensitive to gross outliers in the data:

Example 1.15 The sample mean of the data $[0,0,0,0,0,0,a]$ is $\frac{a}{7}$, which grows without bound as $a \to \infty$. A single gross outlier can thus throw the estimate quite far away. The sample median, however, stays put: it is 0 for all a. ■

The robustness of the spatial median is a consequence of the fact that the generalized Laplace pdf has "heavier tails" than the normal pdf, that is, large deviations are not so rare.

The multivariate Student-t distribution is another "heavy-tail" observation model that can be used for outlier-robust estimation. Matlab/Octave code for estimation of linear models with Student-t noise is available at `http://URN.fi/URN:NBN:fi:tty-201203301089`

2 Random sequences

2.1 Generalities

2.1.1 Statistical specification

The ensemble of a *random sequence* (or: discrete-time random signal) is the set of sequences of scalars or vectors with *index set* (or: time domain, set of epochs) $\mathscr{T}$ which is $\mathbb{Z}_+$ or $\mathbb{N}$ or $\mathbb{Z}$. For each t, $X[t]$ is a rv (length-n vector) whose set of possible values is called the *state space*, denoted $\mathscr{S}$. The state space may be $\mathbb{R}^n$ (continuous state) or a countable subset of $\mathbb{R}^n$ (discrete state).

The probability law of a random sequence is specified by the set of all possible first order probability distributions, all possible second order probability distributions, and so on for all orders. The first order distributions are the cdf's (or pmf's or pdf's) of $X[t]$ for all $t \in \mathscr{T}$; the second order distributions are the cdf's of

$$X[t_1,t_2] = \begin{bmatrix} X[t_1] \\ X[t_2] \end{bmatrix}$$

for all $t_1 < t_2 \in \mathscr{T}$; the third order distributions are the cdf's of $X[t_1,t_2,t_3]$ for all $t_1 < t_2 < t_3 \in \mathscr{T}$; and so on.

The specifications must be *consistent* in the sense that any marginal distribution of a specified higher order distribution should agree with the corresponding specified lower order distribution. For example, if $t_1 < t_2 < t_3$ the marginal cdf's

$$f_{X[t_{1:2}]}(\begin{bmatrix} \infty \\ x[2] \end{bmatrix}) \text{ and } f_{X[t_{2:3}]}(\begin{bmatrix} x[2] \\ \infty \end{bmatrix})$$

should both agree with $f_{X[t_2]}(x[2])$ for every $x[2] \in \mathscr{S}$.

If we talk about two rs's Y and Z, it means that there is a rs $X = \begin{bmatrix} Y \\ Z \end{bmatrix}$ for which probability distributions of all orders can be defined. The rs's Y and Z are *independent* if the joint probability distributions of all orders can be factored, i.e.

$$\begin{aligned} &\mathsf{P}(Y[t_1] \le y[1], \ldots, Y[t_m] \le y[m], Z[t_1] \le z[1], \ldots, Z[t_m] \le z[m]) \\ &= \mathsf{P}(Y[t_1] \le y[1], \ldots, Y[t_m] \le y[m]) \cdot \mathsf{P}(Z[t_1] \le z[1], \ldots, Z[t_m] \le z[m]) \end{aligned}$$

2.1.2 Moments

The *mean sequence* of a random sequence X is the (nonrandom) sequence of n-vectors

$$\mu_X[t] = \mathsf{E}X[t]$$

(provided that the expected values exist). The sequence $\mathsf{var}X[t]$ is defined analogously. The moment sequences $\mu_X[t]$ and $\mathsf{var}X[t]$ are determined by the first-order distribution.

Second order distributions are used to determine the *autocorrelation sequence* $\mathbf{r}_X[t_1,t_2] = \mathsf{E}\big(X[t_1]X'[t_2]\big)$ and the *autocovariance sequence* $\mathbf{c}_X[t_1,t_2] = \mathsf{cov}(X[t_1],X[t_2]) = \mathbf{r}_X[t_1,t_2] - \mu_X[t_1]\mu_X'[t_2]$. Note that $\mathsf{var}X[t] = \mathbf{c}_X[t,t]$.

A *finite-variance* random sequence is one that has finite $\|X[t]\|_{\mathrm{rms}}^2$. If X is a finite-variance rs then the sequences μ_X, $\mathbf{r}_X$, and $\mathbf{c}_X$ exist.

The correlation sequence $\mathbf{r}_X$ is *symmetric* in the sense that $\mathbf{r}_X[t_1,t_2] = \mathbf{r}_X'[t_2,t_1]$. It is also a *nonnegative definite* two-index sequence, which means that for any finite set of epochs $t_1,\ldots,t_m$ and n-vectors $a[1],\ldots,a[m]$, the $m \times m$ matrix

$$\mathbf{z} = \begin{bmatrix} a'[1]\mathbf{r}_X[t_1,t_1]a[1] & \cdots & a'[1]\mathbf{r}_X[t_1,t_m]a[m] \\ \vdots & \ddots & \vdots \\ a'[m]\mathbf{r}_X[t_m,t_1]a[1] & \cdots & a'[m]\mathbf{r}_X[t_m,t_m]a[m] \end{bmatrix}$$

is nonnegative definite.

Proof. For any m-vector α,

$$\alpha'\mathbf{z}\alpha = \mathsf{E}\Big(\sum_{j=1}^{m} \alpha_j a'[j]X[t_j]\Big)^2 \geq 0 \qquad \blacksquare$$

As a consequence of the Cauchy-Schwartz inequality, the autocorrelation sequence satisfies the inequality

$$|\operatorname{tr}\mathbf{r}_X[t_1,t_2]| \leq \sqrt{\operatorname{tr}\mathbf{r}_X[t_1,t_1]} \cdot \sqrt{\operatorname{tr}\mathbf{r}_X[t_2,t_2]}$$

In the case that X is a sequence of scalar-valued rv's, the autocorrelation sequence r_X is a two-index sequence of scalars with the properties: $r_X[t_1,t_2] = r_X[t_2,t_1]$, the $m \times m$ matrix

$$\begin{bmatrix} r_X[t_1,t_1] & \cdots & r_X[t_1,t_m] \\ \vdots & \ddots & \vdots \\ r_X[t_m,t_1] & \cdots & r_X[t_m,t_m] \end{bmatrix}$$

is symmetric nonnegative definite for any $t_1,\ldots,t_m \in \mathscr{T}$, and $|r_X[t_1,t_2]| \leq \sqrt{r_X[t_1,t_1]} \cdot \sqrt{r_X[t_2,t_2]}$.

The autocovariance sequence $\mathbf{c}_X$ is also symmetric, nonnegative definite, and satisfies the inequality $|\operatorname{tr}\mathbf{c}_X[t_1,t_2]| \leq \sqrt{\operatorname{tr}\mathbf{c}_X[t_1,t_1]} \cdot \sqrt{\operatorname{tr}\mathbf{c}_X[t_2,t_2]}$.

Partitioning the vectors of a random sequence according to $X = \begin{bmatrix} Y \\ Z \end{bmatrix}$, we can define the *cross-correlation sequence* $\mathbf{r}_{YZ}[t_1,t_2] = \mathsf{E}\big(Y[t_1]Z'[t_2]\big)$ and the *cross-covariance sequence* $\mathbf{c}_{YZ}[t_1,t_2] = \mathsf{cov}(Y[t_1],Z[t_2]) = \mathbf{r}_{YZ}[t_1,t_2] - \mu_Y[t_1]\mu_Z'[t_2]$. These two-index sequences exist if Y and Z are finite variance random sequences. Random sequences Y,Z are *uncorrelated* if $\mathbf{c}_{YZ}[t_1,t_2] = 0$ for all $t_1,t_2 \in \mathscr{T}$. The cross-correlation sequence has the symmetry property $\mathbf{r}_{YZ}[t_1,t_2] = \mathbf{r}'_{ZY}[t_2,t_1]$ and, if Y and Z have the same length, satisfies the Cauchy-Schwartz inequality

$$|\operatorname{tr}\mathbf{r}_{YZ}[t_1,t_2]| \le \sqrt{\operatorname{tr}\mathbf{r}_Y[t_1,t_1]} \cdot \sqrt{\operatorname{tr}\mathbf{r}_Z[t_2,t_2]}$$

and similarly for the cross-covariance sequence $\mathbf{c}_{YZ}$.

2.1.3 Stationary random sequences

A rs X is said to be *mth order stationary* if all mth order distributions are invariant to a shift of the time origin, that is,

$$f_{X[t_1+\tau:t_m+\tau]}(x[1{:}m]) = f_{X[t_1:t_m]}(x[1{:}m])$$

for all $t_1 < \cdots < t_m \in \mathscr{T}$ and all τ such that $t_1+\tau < \cdots < t_m+\tau \in \mathscr{T}$. Because of consistency, it is then stationary for all lower orders $1,\ldots,m-1$ as well. The rs is *strictly stationary* if it is mth order stationary for all $m \in \mathbb{N}$.

A rs X is said to be *wide-sense stationary* (or: wss, weakly stationary) if it is finite-variance, its mean sequence is constant, and its autocorrelation sequence is index-shift invariant, that is,

$$\mathbf{r}_X[t_1+\tau,t_2+\tau] = \mathbf{r}_X[t_1,t_2]$$

for all $t_1,t_2 \in \mathscr{T}$ and all τ such that $t_1+\tau,t_2+\tau \in \mathscr{T}$. Because the autocorrelation sequence and autocovariance sequence of a wss random sequence depend only on the difference between the indices, we can use one-index notation:

$$\mathbf{r}_X[\tau] := \mathbf{r}_X[t+\tau,t],\ \mathbf{c}_X[\tau] := \mathbf{c}_X[t+\tau,t] \qquad (t,t+\tau \in \mathscr{T})$$

A finite-variance random sequence that is second order stationary is wide-sense stationary. The following example shows that the converse is not true.

> *Example 2.1.* Let $X[t]$ be a scalar-valued random sequence with index set $\mathbb{Z}$. Assume that $X[t_1{:}t_m]$ are mutually independent rv's for all $t_1 < \cdots < t_m \in \mathbb{Z}$, and that $X[t] \sim \text{Normal}(0,1)$ when t is even and $X[t]+2 \sim \text{Categorical}(\frac{1}{2},0,\frac{1}{2})$ when t is odd. Clearly, X is not first-order stationary. However, it is wss, because $\mu_X[t] = 0$ and
>
> $$r_X[t_1,t_2] = 1_{\{0\}}(t_2-t_1) =: \delta(t_1-t_2) \qquad \blacksquare$$

The autocorrelation sequence $\mathbf{r}_X[\cdot]$ of a wss sequence is symmetric in the sense that $\mathbf{r}_X[\tau] = \mathbf{r}'_X[-\tau]$, and is nonnegative definite in the sense that

$$\begin{bmatrix} a'[1]\mathbf{r}_X[0]a[1] & \cdots & a'[1]\mathbf{r}'_X[\tau_m]a[m] \\ \vdots & \ddots & \vdots \\ a'[m]\mathbf{r}_X[\tau_m]a[1] & \cdots & a'[m]\mathbf{r}_X[0]a[m] \end{bmatrix}$$

is nonnegative definite for any $\tau_2, \ldots, \tau_m$ and n-vectors $a[1], \ldots, a[m]$. Also, by the Cauchy-Schwartz inequality, the autocorrelation sequence satisfies the inequality

$$|\operatorname{tr}\mathbf{r}_X[\tau]| \leq \operatorname{tr}\mathbf{r}_X[0] \; (= \|X\|^2_{\text{rms}})$$

If $\operatorname{tr}\mathbf{r}_X[\tau_1] = \operatorname{tr}\mathbf{r}_X[0]$ for some τ_1 then $\operatorname{tr}\mathbf{r}_X[t]$ is τ_1-periodic.

Proof.

$$\begin{aligned} |\operatorname{tr}\mathbf{r}_X[\tau+\tau_1] - \operatorname{tr}\mathbf{r}_X[\tau]|^2 &= \left|\mathsf{E}X'[t](X[t+\tau+\tau_1] - X[t+\tau])\right|^2 \\ &\overset{\text{cs}}{\leq} \|X[t]\|^2_{\text{rms}} \cdot \|X[t+\tau+\tau_1] - X[t+\tau]\|^2_{\text{rms}} \\ &= \operatorname{tr}\mathbf{r}_X[0] \cdot 2(\underbrace{\operatorname{tr}\mathbf{r}_X[0] - \operatorname{tr}\mathbf{r}_X[\tau_1]}_{=0}) \qquad \blacksquare \end{aligned}$$

In the case of a scalar rs, the properties can be stated more simply: $r_X[\tau] = r_X[-\tau]$, the $m \times m$ matrix

$$\begin{bmatrix} r_X[0] & \cdots & r_X[\tau_m] \\ \vdots & \ddots & \vdots \\ r_X[\tau_m] & \cdots & r_X[0] \end{bmatrix}$$

is symmetric nonnegative definite for any $\tau_2, \ldots, \tau_m$, and $|r_X[\tau]| \leq r_X[0] \; (= \|X\|^2_{\text{rms}})$.

Similarly, the autocovariance sequence $\mathbf{c}_X[\cdot]$ is symmetric, nonnegative definite, and satisfies the inequality $|\operatorname{tr}\mathbf{c}_X[\tau]| \leq \operatorname{tr}\mathbf{c}_X[0] \; (= \operatorname{tr}\mathsf{var}X)$.

2.2 Some basic random sequences

2.2.1 Random constant sequence

Let A be a scalar rv and consider the random sequence

$$X[t] = A \quad (t \in \mathbb{Z})$$

For this rs, any sample path (or: realization) is a sequence whose values are all equal to the same value $a \in \mathscr{E}_A$.

The mth order cdf of this random sequence is

$$\begin{aligned} &f_{X[t_{1:m}]}(x[1{:}m]) = \mathsf{P}(X[t_1] \leq x[1], \ldots, X[t_m] \leq x[m]) \\ &= \mathsf{P}(A \leq x[1], \ldots, A \leq x[m]) = \mathsf{P}(A \leq \min(x[1{:}m])) = f_A(\min(x[1{:}m])) \end{aligned}$$

This specification is consistent: for example, marginalising $X[t_2]$ out of $f_{X[t_{1:2}]}$ gives

$$f_{X[t_{1:2}]}\left(\begin{bmatrix} x[1] \\ \infty \end{bmatrix}\right) = f_A\left(\min \begin{bmatrix} x[1] \\ \infty \end{bmatrix}\right) = f_A(x[1]) = f_{X[t_1]}(x[1])$$

Note that if A is a continuous rv then so are the first order distributions of $X[\cdot]$, but higher order distributions are not continuous.

The mean sequence is $\mu_X[t] = \mathsf{E}X[t] = \mathsf{E}A$. The autocorrelation sequence is

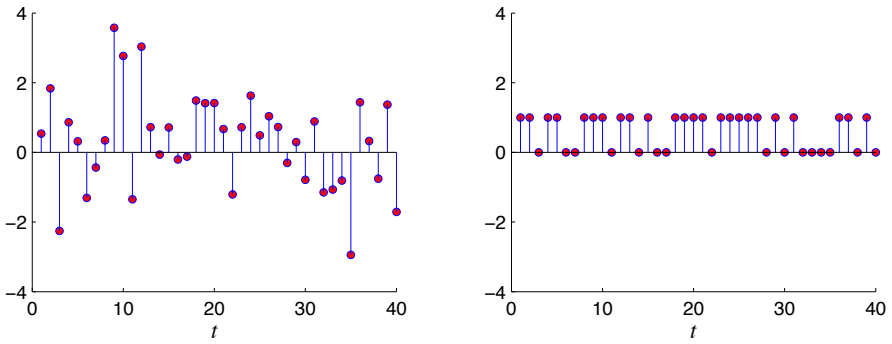
$$r_X[t_1,t_2] = \mathsf{E}X[t_1]X[t_2] = \mathsf{E}(A^2)$$

and the autocovariance sequence is $c_X[t_1,t_2] = \mathsf{var}A$. The rs is strictly stationary, so these could also be written $r_X[\tau] = \mathsf{E}(A^2)$ and $c_X[\tau] = \mathsf{var}A$.

2.2.2 IID sequence and white noise

A basic building block for modelling with random sequences is the *iid random sequence*. Its specification is: Every $X[t]$ has the same first-order distribution, and the rv's in any set $\{X[t_1],\ldots,X[t_m] : t_1 < \cdots < t_m\}$ are mutually independent. An iid rs is strictly stationary.

In the following figure, the left plot shows a portion of a sample path of a scalar iid sequence with standard normal distribution; the right plot is a portion of a sample path of a scalar iid sequence with Bernoulli($\frac{1}{2}$) distribution.



Note that a "random constant" Normal$(0,1)$ sequence has the same first order distribution as an iid Normal$(0,1)$ sequence, but typical sample paths look quite different!

The mean sequence of an iid sequence is the constant $\mathsf{E}X[t_0]$, where t_0 is any index in $\mathscr{T}$. The autocovariance sequence is

$$\begin{aligned} c_X[t_1,t_2] &= \mathsf{cov}(X[t_1],X[t_2]) = \mathsf{cov}(X[t_0+t_1-t_2],X[t_0]) \\ &= \big(\mathsf{var}X[t_0]\big)\cdot\delta(t_1-t_2) = \begin{cases} \mathsf{var}X[t_0] & \text{if } t_1 = t_2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

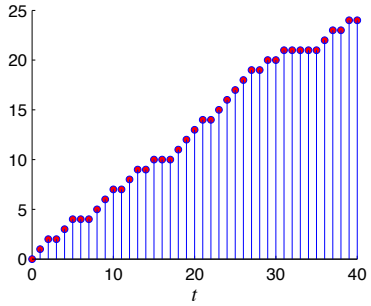
A zero-mean wss rs that has an autocovariance sequence of the form $\mathbf{c}_X[t] = \mathbf{p}\delta(t)$ is called (discrete) *white noise*. Any zero-mean iid sequence is white noise. An iid sequence whose first-order distribution is normal is called Gaussian white noise.

2.2.3 Binomial sequence

A *binomial sequence* (or: binomial process, Bernoulli counting process) is constructed from an iid Bernoulli sequence $X[\cdot]$ according to

$$S[t] = \sum_{k=1}^{t} X[k] \quad (t \in \mathbb{Z}_+)$$

Here is the sample path of the binomial sequence that is obtained by taking a running sum of the Bernoulli sample path that was shown earlier:



Using the results of Example 1.11 in §1.3.6, the first order distribution of a Binomial sequence is $S[t] \sim \text{Binomial}(\theta, t)$. Note that the distribution is finite: $p_{S[t]}(s) = 0$ for $s \notin \mathbb{Z}_{t+1} = \{0, \ldots, t\}$. The second order pmf is

$$\begin{aligned} p_{S[t_{1:2}]}(s[1{:}2]) &= \mathsf{P}(S[t_1] = s[1], S[t_2] = s[2]) \\ &= \mathsf{P}(S[t_1] = s[1], S[t_2] - S[t_1] = s[2] - s[1]) \\ &= \mathsf{P}(X[1] + \ldots + X[t_1] = s[1], X[t_1 + 1] + \ldots + X[t_2] = s[2] - s[1]) \\ &= \mathsf{P}(S[t_1] = s[1]) \cdot \mathsf{P}(S[t_2] - S[t_1] = s[2] - s[1]) \end{aligned}$$

Thus, the second order pmf $p_{S[t_{1:2}]}(s[1{:}2])$ is the product of two binomial pmf's, namely the pmf of $S[t_1] \sim \text{Binomial}(\theta, t_1)$ and the pmf of $S[t_2] - S[t_1] \sim \text{Binomial}(\theta, t_2 - t_1)$, that is,

$$p_{S[t_{1:2}]}(s[1{:}2]) = \binom{t_1}{s[1]} \theta^{s[1]} (1-\theta)^{t_1 - s[1]} \cdot \binom{t_2 - t_1}{s[2] - s[1]} \theta^{s[2]-s[1]} (1-\theta)^{t_2 - t_1 - (s[2]-s[1])}$$

Higher order pmf's can be constructed analogously.

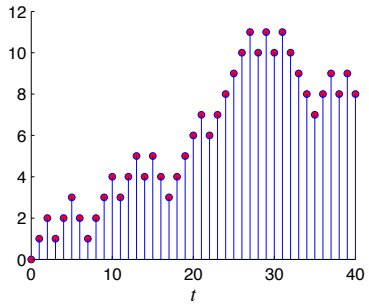
The mean and variance of the binomial sequence are $\mathsf{E}S[t] = t\theta$ and $\mathsf{var}S[t] = t\theta(1-\theta)$. The rs is thus not stationary in any sense.

2.2.4 Random walk

A (Bernoulli symmetric) random walk is the random sequence that is constructed from an iid Bernoulli($\frac{1}{2}$) sequence $X[\cdot]$ or from the corresponding Binomial sequence $S[\cdot]$ according to

$$V[t] = \sum_{k=1}^{t} (2X[k] - 1) = 2S[t] - t \quad (t \in \mathbb{Z}_+)$$

It models the position of a particle that starts at the origin and, at each epoch, takes a step forward or backward with equal probability. Here is the sample path of the random walk that is obtained from the Bernoulli sample path that was shown earlier:



Its first order pmf is

$$\begin{aligned} p_{V[t]}(v) &= \mathsf{P}(V[t]=v) = \mathsf{P}(2S[t]-t=v) = \mathsf{P}(S[t]=\frac{t+v}{2}) \\ &= \binom{t}{(t+v)/2}\frac{1}{2^t}1_{\{-t,-t+2,\ldots,t\}}(v) \end{aligned}$$

Second and higher order distributions can be found by using the "independent increments" property, as was done in §2.2.3 for the Binomial sequence.

The mean and variance of the random walk are $\mathsf{E}V[t]=0$ and $\mathsf{var}V[t]=t$. When $t_1 \le t_2$ we have

$$\begin{aligned} \mathsf{E}(V[t_1]V[t_2]) &= \mathsf{E}\big(V[t_1]^2 + V[t_1](V[t_2]-V[t_1])\big) \\ &= \mathsf{E}(V[t_1]^2) + \mathsf{E}\big(V[t_1](V[t_2]-V[t_1])\big) \\ &= \mathsf{E}(V[t_1]^2) + \big(\mathsf{E}V[t_1]\big)\cdot\big(\mathsf{E}(V[t_2]-V[t_1])\big) \\ &= t_1 \end{aligned}$$

and similarly $\mathsf{E}(V[t_1]V[t_2]) = t_2$ when $t_2 \le t_1$. Thus

$$r_V[t_1,t_2] = c_V[t_1,t_2] = \min(t_1,t_2)$$

The rs is not stationary in any sense.

2.2.5 Autoregressive sequence

A scalar *autoregressive sequence* is constructed from a white noise sequence $X[\cdot]$ having $\mathsf{var}X[t]=\sigma^2$ according to the difference equation

$$Y[t] = \alpha Y[t-1] + X[t] \quad (t \in \mathbb{N})$$

The initial value is denoted $Y[0]=\gamma A$; it is assumed that $\mathsf{E}A=0$, $\mathsf{var}A=1$, and $\mathsf{cov}(A,X[t])=0$ for all $t \in \mathbb{N}$. Then

$$Y[t] = \alpha^t \gamma A + \sum_{k=1}^{t} \alpha^{t-k} X[k]$$

The AR sequence is zero-mean, because

$$\mathsf{E}Y[t] = \alpha^t \gamma \mathsf{E}A + \sum_{k=1}^{t} \alpha^{t-k} \mathsf{E}X[k] = 0$$

The AR sequence is uncorrelated to later inputs, that is, $\mathsf{cov}(Y[t_1], X[t_2]) = 0$ for all $0 \le t_1 < t_2$.

> *Proof.* Induction on t_1. By the specification of the sequence, $Y[0]$ is uncorrelated to $X[t]$ for all $t > 0$. Suppose that for some $t_1 \ge 0$, $\mathsf{cov}(Y[t_1], X[t]) = 0$ for all $t > t_1$. Then for any $t_2 > t_1 + 1$ we have
>
> $$\mathsf{E}Y[t_1+1]X[t_2] = \mathsf{E}(\alpha Y[t_1] + X[t_1+1])X[t_2] = 0 \qquad \blacksquare$$

The AR sequence $\mathsf{var}Y[\cdot]$ follows the recursion

$$\mathsf{var}Y[t] = \mathsf{var}(\alpha Y[t-1]) + \mathsf{var}X[t] = \alpha^2 \mathsf{var}Y[t-1] + \sigma^2$$

with $\mathsf{var}Y[0] = \gamma^2$. Solving this recursion gives

$$\mathsf{var}Y[t] = \alpha^{2t}\gamma^2 + \sigma^2 \sum_{k=0}^{t-1} \alpha^{2k} = \alpha^{2t}\left(\gamma^2 - \frac{\sigma^2}{1-\alpha^2}\right) + \frac{\sigma^2}{1-\alpha^2}$$

The autocorrelation sequence $r_Y[t+\tau, t]$ for $\tau \ge 0$ is

$$\mathsf{E}Y[t+\tau]Y[t] = \mathsf{E}\left(\alpha^\tau Y[t] + \sum_{k=1}^{\tau} \alpha^{\tau-k} X[k+t]\right) Y[t] = \alpha^\tau \mathsf{var}Y[t]$$

while for $-t \le \tau \le 0$ it is

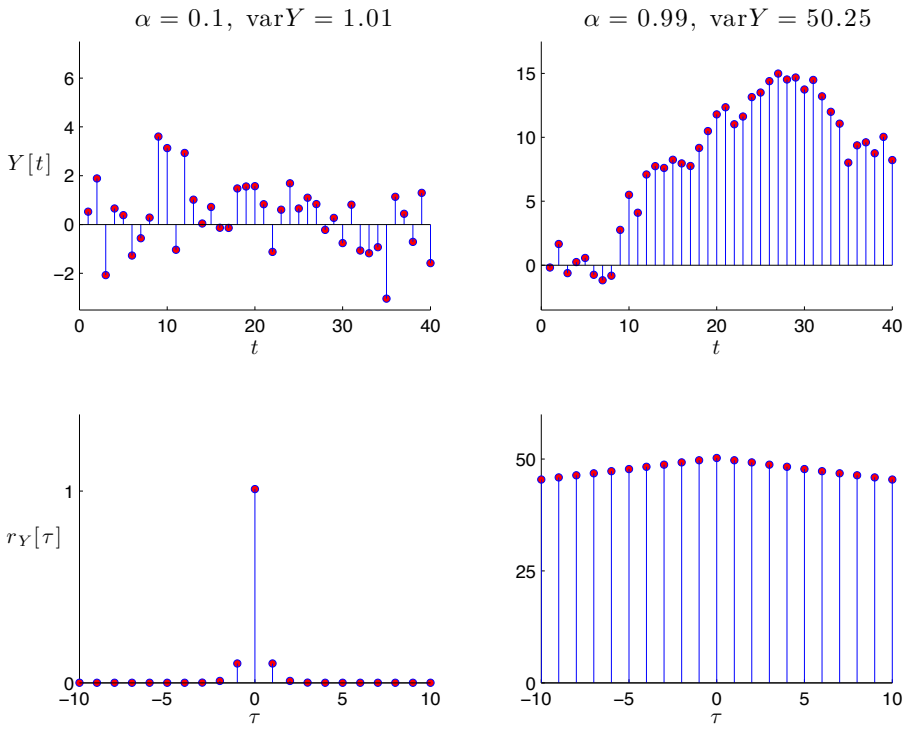
$$r_Y[t+\tau, t] = r_Y[t, t+\tau] = r_Y[t+\tau-\tau, t+\tau] = \alpha^{-\tau}\mathsf{var}Y[t+\tau]$$

If $\gamma = \frac{\pm\sigma}{\sqrt{1-\alpha^2}}$, the AR sequence is wss, because its variance is constant and its autocorrelation is shift-invariant:

$$r_Y[t+\tau, t] = \frac{\alpha^{|\tau|}\sigma^2}{1-\alpha^2} \quad (t, t+\tau \in \mathbb{Z}_+)$$

If $\gamma \neq \frac{\pm\sigma}{\sqrt{1-\alpha^2}}$, the AR sequence is not wss, but if $|\alpha| < 1$ then it is *asymptotically wss*, in the sense that, for any fixed τ, the autocorrelation sequence $r_Y[t+\tau, t]$ converges as $t \to \infty$ to $\frac{\alpha^{|\tau|}\sigma^2}{1-\alpha^2}$.

Here are sample paths of two wss AR sequences, both computed using the Gaussian white noise sample path shown in §2.2.2 and with $A = -0.1022$.



2.3 Convergence

In this section we look at some of the ways that random sequences can converge: convergence in distribution, convergence in probability, convergence in mean square, and convergence with probability one (almost sure convergence). The different convergence modes are related as follows:

$$\begin{array}{ccc} \text{ms} & \searrow & \\ & & \text{p} \ \Rightarrow \ \text{d} \\ \text{as} & \nearrow & \end{array}$$

2.3.1 Convergence in distribution

A rs $X[\cdot]$ is said to *converge in distribution* (or: converge weakly) to a rv A (denoted $X[t] \xrightarrow[t\to\infty]{\text{d}} A$ or $X \xrightarrow{\text{d}} A$) if

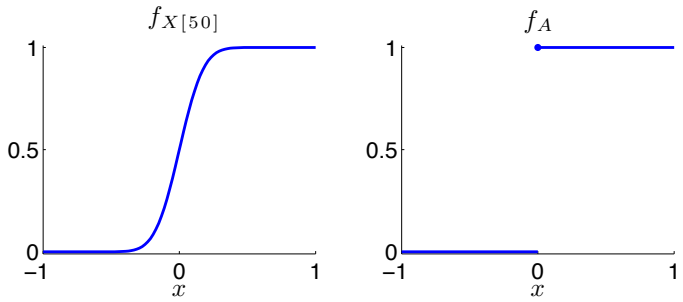
$$\lim_{t\to\infty} f_{X[t]}(x) = f_A(x)$$

at all points x where f_A is continuous.

Example 2.2. The cdf of the random sequence $X[t] \sim \text{Normal}(0, \frac{1}{t})$ with index $t \in \mathbb{N}$ converges towards

$$\lim_{t\to\infty} f_{X[t]}(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

It therefore converges in distribution to the degenerate rv $A \stackrel{\text{as}}{=} 0$, which has cdf $f_A(x) = 1_{[0,\infty)}(x)$.



Note that $\lim_{t\to\infty} f_{X[t]}(0) = \frac{1}{2} \neq 1 = f_A(0)$, but this is ok because $x = 0$ is a discontinuity point of f_A. ■

Because convergence in distribution is defined entirely in terms of the first order distribution, this kind of convergence does not tell you much about the evolution of typical sample paths.

Example 2.3. Consider the rs $X[t] \stackrel{\text{iid}}{\sim} \text{Bernoulli}(\frac{1}{2})$, which obviously converges in distribution to $A = \text{Bernoulli}(\frac{1}{2})$. However, this convergence does not indicate that the sequence is eventually settling down to a constant value as $t \to \infty$. Indeed, typical sample paths of $X[\cdot]$ exhibit "noisy" variations for all t. ■

Lévy's continuity theorem says that if $\lim_{t\to\infty} \phi_{X[t]}(\zeta)$ exists for all ζ and is continuous at $\zeta = 0$ then $X[\cdot]$ converges in distribution to a random variable A whose characteristic function is

$$\phi_A(\zeta) = \lim_{t\to\infty} \phi_{X[t]}(\zeta)$$

A famous instance of convergence in distribution is the *central limit theorem* (CLT), one version of which is: If $X[\cdot]$ is an iid sequence with mean μ and variance $\mathbf{c}$ then

$$S[t] = \frac{1}{\sqrt{t}} \sum_{k=1}^{t} (X[k] - \mu)$$

converges in distribution to a Normal$(0, \mathbf{c})$ rv.

Proof. The characteristic function of $Y[k] = X[k] - \mu$ has the Taylor expansion

$$\phi_{Y[k]}(\zeta) = 1 - \tfrac{1}{2}\zeta'\mathbf{c}\zeta + o(\|\zeta\|^2)$$

The characteristic function of $S[t] = \dfrac{1}{\sqrt{t}} \displaystyle\sum_{k=1}^{t} Y[k]$ is then

$$\phi_{S[t]}(\zeta) = \mathsf{E}\exp(\frac{\imath\zeta'}{\sqrt{t}} \sum_{k=1}^{t} Y[k]) = \prod_{k=1}^{t} \mathsf{E}\exp(\frac{\imath\zeta' Y[k]}{\sqrt{t}}) = \prod_{k=1}^{t} \phi_{Y[k]}(\frac{\zeta}{\sqrt{t}})$$

$$= \left(1 - \frac{1}{2t}\zeta'\mathbf{c}\zeta + o(\frac{\|\zeta\|^2}{t})\right)^t \quad \xrightarrow[t\to\infty]{} \quad \exp(-\tfrac{1}{2}\zeta'\mathbf{c}\zeta)$$

which is the characteristic equation of a Normal$(0, \mathbf{c})$ rv. ■

As a consequence of the CLT, the scaled random walk $\frac{1}{\sqrt{t}} \sum_{k=1}^{t} (2X[k]-1)$, where X is an iid Bernoulli$(\frac{1}{2})$ sequence, converges in distribution to a standard normal rv.

The CLT is often used to derive normal approximations of probability distributions. A particular case is the DeMoivre-Laplace approximation: because $S[t] \sim$ Binomial(θ, t) is the sum of t iid Bernoulli(θ) rv's, the distribution of $S[t]$ for large t is approximately Normal$(t\theta, t\theta(1-\theta))$.

2.3.2 Convergence in probability

A rs $X[\cdot]$ is said to *converge in probability* to a rv A (denoted $X[t] \xrightarrow[t\to\infty]{\text{p}} A$ or $X \xrightarrow{\text{p}} A$) if

$$\lim_{t\to\infty} \mathsf{P}(\|X[t]-A\| > \varepsilon) = 0$$

for all $\varepsilon > 0$. An intuitive interpretation is that, as t grows, the fraction of sample paths of $X[t]-A$ that are outside the origin-centred ball of radius ε decreases to 0.

Example 2.4. Consider the iid rs $X[t] = t \cdot 1_{[0,\frac{1}{t}]}(U)$, where $U \sim$ Uniform$([0,1])$. For any $\varepsilon > 0$ we have $\mathsf{P}(|X[t]| > \varepsilon) = \frac{1}{t} \to 0$. Thus $X \xrightarrow{\text{p}} 0$. ■

Example 2.5 Let $T[k] \sim$ Categorical$(2^{-k}, \ldots, 2^{-k})$, and let

$$X[1] = 1$$
$$X[2] = 1_{\{1\}}(T[1]),\ X[3] = 1_{\{2\}}(T[1])$$
$$X[4] = 1_{\{1\}}(T[2]),\ X[5] = 1_{\{2\}}(T[2]),\ X[6] = 1_{\{3\}}(T[2]),\ X[7] = 1_{\{4\}}(T[2])$$

and so on. Thus, the sequence is mostly zeros, with a 1 occurring at $t = 1$, the next 1 occurring within the next 2 epochs, the next 1 occurring within the next 4 epochs, and so on. The probability that $|X[t]| > 0$ is $2^{-\lfloor \log_2 t \rfloor}$, and so $X \xrightarrow{\text{p}} 0$. ■

Convergence in probability implies convergence in distribution.

Proof (scalar case): For any x and $\varepsilon > 0$ we have

$$f_{X[t]}(x) = \mathsf{P}(X[t] \le x) = \mathsf{P}(X[t] \le x,\ A \le x+\varepsilon) + \mathsf{P}(X[t] \le x,\ A > x+\varepsilon)$$

and

$$f_A(x+\varepsilon) = \mathsf{P}(A \le x+\varepsilon) = \mathsf{P}(A \le x+\varepsilon,\ X[t] \le x) + \mathsf{P}(A \le x+\varepsilon,\ X[t] > x)$$

and so

$$\begin{aligned}
|f_{X[t]}(x) - f_A(x+\varepsilon)| &\leq \mathsf{P}(X[t] \leq x,\, A > x+\varepsilon) + \mathsf{P}(A \leq x+\varepsilon,\, X[t] > x) \\
&= \mathsf{P}(X[t] \leq x,\, A > x+\varepsilon) + \mathsf{P}(A \leq x-\varepsilon,\, X[t] > x) \\
&\quad + \mathsf{P}(x-\varepsilon < A \leq x+\varepsilon,\, X[t] > x) \\
&\leq \mathsf{P}(A - X[t] > \varepsilon) + \mathsf{P}(X[t] - A > \varepsilon) \\
&\quad + \mathsf{P}(x-\varepsilon < A \leq x+\varepsilon) \\
&= \mathsf{P}(|X[t] - A| > \varepsilon) + \mathsf{P}(x-\varepsilon < A \leq x+\varepsilon)
\end{aligned}$$

Suppose $X \xrightarrow{\mathrm{p}} A$, and let x be a continuity point of f_A. Given any $\delta > 0$, one can choose $\varepsilon > 0$ such that $\mathsf{P}(x-\varepsilon < A \leq x+\varepsilon) < \frac{\delta}{3}$ and $f_A(x+\varepsilon) - f_A(x) < \frac{\delta}{3}$, then choose t large enough so that $\mathsf{P}(|X[t] - A| > \varepsilon) < \frac{\delta}{3}$. Then

$$|f_{X[t]}(x) - f_A(x)| \leq |f_{X[t]}(x) - f_A(x+\varepsilon)| + |f_A(x+\varepsilon) - f_A(x)| < \delta$$

Thus $\lim_{t\to\infty} f_{X[t]}(x) = f_A(x)$. ■

The following Cauchy-type convergence criterion is useful because it does not require knowledge of the limit: X converges in probability iff

$$\lim_{t_1,t_2\to\infty} \mathsf{P}(\|X[t_2] - X[t_1]\| > \varepsilon) = 0$$

for any $\varepsilon > 0$. In other words, for any $\delta, \varepsilon > 0$ there exists a t_δ such that $\mathsf{P}(\|X[t_2] - X[t_1]\| > \varepsilon) < \delta$ for all $t_1, t_2 > t_\delta$.

The Cauchy criterion is used in the following example, which shows that convergence in distribution does not imply convergence in probability.

Example 2.3. (continued) For any $t_2 > t_1$ the pmf of $X[t_2] - X[t_1]$ is

$$\mathsf{P}(X[t_2] - X[t_1] = x) = \begin{cases} \frac{1}{4} & x = -1 \\ \frac{1}{2} & x = 0 \\ \frac{1}{4} & x = 1 \end{cases}$$

and so $\mathsf{P}(|X[t_2] - X[t_1]| > \varepsilon) = \frac{1}{2}$ for any $\varepsilon < 1$. Thus X does not converge in probability. ■

2.3.3 Convergence in mean square

A rs $X[\cdot]$ is said to *converge in mean square* to a rv A (denoted $X[t] \xrightarrow[t\to\infty]{\mathrm{ms}} A$ or $X \xrightarrow{\mathrm{ms}} A$ or $\operatorname*{l.i.m}_{t\to\infty} X[t] = A$) if X is a finite variance sequence and

$$\lim_{t\to\infty} \|X[t] - A\|_{\mathrm{rms}} = 0$$

Example 2.5. (continued)

$$\|X[t]\|_{\text{rms}}^2 = \mathsf{E}|X[t]|^2 = \mathsf{P}(X[t]=1) = 2^{-\lfloor \log_2 t \rfloor} \to 0$$

and so $X \overset{\text{ms}}{\to} 0$. ■

If $X[\cdot]$ is a wss rs with uncorrelated sequence terms (i.e. $\mathsf{cov}(X[t_1],X[t_2]) = 0$ for $t_1 \neq t_2$) then $\frac{1}{t}\sum_{k=1}^{t} X[k] \xrightarrow[t\to\infty]{\text{ms}} \mu_X$.

Proof. First, note that

$$\mathsf{E}\left(\frac{1}{t}\sum_{k=1}^{t} X[k]\right) = \frac{1}{t}\sum_{k=1}^{t} \mathsf{E}X[k] = \frac{1}{t}\sum_{k=1}^{t} \mu_X = \mu_X$$

Then, denoting $\mathsf{var}X[t] = \mathbf{c}$, we have

$$\|\big(\frac{1}{t}\sum_{k=1}^{t} X[k]\big) - \mu\|_{\text{rms}}^2 = \text{tr}\,\mathsf{var}\left(\frac{1}{t}\sum_{k=1}^{t} X[k]\right) = \text{tr}\,\frac{1}{t^2}\mathsf{var}\big(\sum_{k=1}^{t} X[k]\big)$$

$$= \text{tr}\,\frac{1}{t^2}\sum_{k=1}^{t} \mathsf{var}X[k] = \frac{1}{t^2}\sum_{k=1}^{t} \text{tr}\,\mathsf{var}X[k] = \frac{1}{t^2}(t\,\text{tr}\,\mathbf{c}) \to 0 \qquad ■$$

Because of Chebyshev's inequality

$$\mathsf{P}(\|X[t]-A\| > \varepsilon) \leq \frac{\|X[t]-A\|_{\text{rms}}^2}{\varepsilon^2}$$

convergence in mean square implies convergence in probability, and thus also convergence in distribution.

In particular, the above result implies the *weak law of large numbers*: If $X[\cdot]$ is an uncorrelated wss rs then $\frac{1}{t}\sum_{k=1}^{t} X[k] \xrightarrow[t\to\infty]{\text{p}} \mu_X$.

The Cauchy criterion for mean square convergence is

$$\lim_{t_1,t_2\to\infty} \|X[t_1]-X[t_2]\|_{\text{rms}} = 0$$

In other words, for any $\delta > 0$ there exists a d such that $\|X[t_2]-X[t_1]\|_{\text{rms}} < \delta$ for all $t_1, t_2 > d$.

The Cauchy criterion is used in the following example, which shows that convergence in probability does not imply convergence in mean square.

Example 2.4. (continued)

$$\|X[t_2]-X[t_1]\|_{\text{rms}} \geq \big|\|X[t_2]\|_{\text{rms}} - \|X[t_1]\|_{\text{rms}}\big| = |\sqrt{t_2}-\sqrt{t_1}|$$

and so the finite-variance rs $X[\cdot]$ does not converge in mean square.
■

Because $\mathsf{var}(X[t]-A)$ is nonnegative definite, it follows that

$$\|\mathsf{E}X[t]-\mathsf{E}A\|^2 \le \|X[t]-A\|^2_{\text{rms}}$$

and so $X \xrightarrow{\text{ms}} A$ implies $\lim_{t\to\infty} \mathsf{E}X[t] = \mathsf{E}A$, that is, expectation and limit operators commute in the sense that

$$\lim_{t\to\infty} \mathsf{E}X[t] = \mathsf{E}(\text{l.i.m.}_{t\to\infty} X[t])$$

Similarly, if $X \xrightarrow{\text{ms}} A$ and $Y \xrightarrow{\text{ms}} B$ then $\lim_{s,t\to\infty} \mathsf{E}(X[t]Y'[s]) = \mathsf{E}(AB')$.

Proof:

$$\begin{aligned}
|\mathsf{E}(X_i[t]Y_j[s]) - \mathsf{E}(A_iB_j)| &= |\mathsf{E}((X_i[t]-A_i)Y_j[s]) + \mathsf{E}(A_i(Y_j[s]-B_j))| \\
&\le |\mathsf{E}((X_i[t]-A_i)Y_j[s])| + |\mathsf{E}(A_i(Y_j[s]-B_j))| \\
&\le \|X_i[t]-A_i\|_{\text{rms}} \cdot \|Y_j[s]\|_{\text{rms}} + \|A_i\|_{\text{rms}} \cdot \|Y_j[s]-B_j\|_{\text{rms}} \\
&\le \underbrace{\|X_i[t]-A_i\|_{\text{rms}}}_{\to 0} \cdot (\underbrace{\|Y_j[s]-B_j\|_{\text{rms}}}_{\to 0} + \|B_j\|_{\text{rms}}) \\
&\quad + \|A_i\|_{\text{rms}} \cdot \underbrace{\|Y_j[s]-B_j\|_{\text{rms}}}_{\to 0} \qquad \blacksquare
\end{aligned}$$

From this we can derive the Loève criterion for MS convergence: a rs X converges in mean square iff there is a constant matrix $\mathbf{r}$ such that $\lim_{s,t\to\infty} \mathsf{E}(X[t]X'[s]) = \mathbf{r}$.

Proof: If $X \xrightarrow{\text{ms}} A$ then (by the previous result) $\lim_{s,t\to\infty} \mathsf{E}(X[t]X'[s]) = \mathsf{E}(AA')$. Conversely, if $\lim_{s,t\to\infty} \mathsf{E}(X[t]X'[s]) = \mathbf{r}$ then

$$\begin{aligned}
\|X[t]-X[s]\|^2_{\text{rms}} &= \text{tr}\,\mathsf{E}(X[t]-X[s])(X[t]-X[s])' \\
&= \text{tr}(\mathsf{E}X[t]X'[t] - \mathsf{E}X[t]X'[s] - \mathsf{E}X[s]X'[t] + \mathsf{E}X[s]X'[s]) \\
&\to \text{tr}(\mathbf{r}-\mathbf{r}-\mathbf{r}+\mathbf{r}) = 0 \qquad \blacksquare
\end{aligned}$$

If each $X[t]$ is a normal random variable and $X \xrightarrow{\text{ms}} A$ then A is a normal rv.

Proof. If $X \xrightarrow{\text{ms}} A$ then $X \xrightarrow{\text{d}} A$, and so

$$\begin{aligned}
\phi_A(\zeta) &= \lim_{t\to\infty} \phi_{X[t]}(\zeta) = \lim_{t\to\infty} \mathrm{e}^{\imath\zeta'\mathsf{E}X[t] - \frac{1}{2}\zeta'(\mathsf{var}X[t])\zeta} \\
&= \mathrm{e}^{\imath\zeta'(\lim_{t\to\infty}\mathsf{E}X[t]) - \frac{1}{2}\zeta'(\lim_{t\to\infty}\mathsf{var}X[t])\zeta} = \mathrm{e}^{\imath\zeta'(\mathsf{E}A) - \frac{1}{2}\zeta'(\mathsf{var}A)\zeta} \qquad \blacksquare
\end{aligned}$$

2.3.4 Convergence with probability one

A rs $X[\cdot]$ is said to *converge with probability one* (or: almost surely, stochastically, pathwise) to a rv A (denoted $X[t] \xrightarrow[t\to\infty]{\text{as}} A$ or $X \xrightarrow{\text{as}} A$) if

$$\mathsf{P}(\lim_{t\to\infty} X[t] = A) = 1$$

In other words, for almost any sample path $x[\cdot]$ of $X[\cdot]$ and corresponding sample a of A (that is, for any non-zero-probability event $(x[\cdot], a)$), we have $x[t] \to a$. A special case of probability-one convergence is when $x[t] \to a$ for *all* sample paths.

The following example shows that probability-one convergence does not imply mean square convergence.

> *Example 2.4.* (continued) The only sample path that is not eventually zero is the sample path corresponding to $U = 0$, and this outcome has zero probability. Thus $X \xrightarrow{\text{as}} 0$. ■

Convergence with probability one implies convergence in probability and hence convergence in distribution.

A famous instance of convergence with probability one is the *strong law of large numbers*: If $X[\cdot]$ is an iid finite-variance rs then $\frac{1}{t}\sum_{k=1}^{t} X[k] \xrightarrow[t\to\infty]{\text{as}} \mu_X$.

> *Example 2.6.* Consider a rs $X[t] \overset{\text{iid}}{\sim} \text{Bernoulli}(\frac{1}{2})$ (e.g. a sequence of coin tosses). A string of 100 consecutive zeros is very unlikely, but it is not impossible; in fact, this string (or any other given finite-length string) appears infinitely often in almost all sample paths!
>
> To show this, introduce the rs
>
> $$Y[t] = \begin{cases} 1 & \text{if } X[100(t-1)+1{:}100] \text{ are all zeros} \\ 0 & \text{otherwise} \end{cases}$$
>
> This is an iid Bernoulli(2^{-100}) random sequence. By the strong law of large numbers, $\lim_{t\to\infty} \frac{1}{t}\sum_{k=1}^{t} y[k] = 2^{-100}$ for almost any sample path $y[\cdot]$. Thus, there is some t_1 such that $\frac{1}{t}\sum_{k=1}^{t} y[k] > 2^{-101}$ for all $t \geq t_1$. Then, for any $m > 0$, taking $t = \max(t_1, 2^{101}m)$ gives $\sum_{k=1}^{t} y[k] \geq m$, that is, the string occurs at least m times. ■

A Cauchy criterion for probability-one convergence is that

$$\lim_{t_1,t_2\to\infty} \|x[t_2] - x[t_1]\| = 0$$

for almost all sample paths of $X[\cdot]$.

The Cauchy criterion is used in the following example, which shows that convergence in probability does not imply probability-one convergence, and also that mean square convergence does not imply probability-one convergence.

> *Example 2.5.* (continued) For any sample path and any $t_1 > 1$, there is a t_2 lying in the same epoch group (i.e. $\lfloor \log_2 t_1 \rfloor = \lfloor \log_2 t_2 \rfloor$) such that $|x[t_2] - x[t_1]| = 1$. Thus $|x[t_2] - x[t_1]|$ converges to zero for *none* of the sample paths, and so $X[\cdot]$ does not exhibit probability-one convergence. ■

We have now seen an example showing that probability-one convergence does not imply mean square convergence, and an example showing that mean square convergence does not imply probability-one convergence. However, if $X \xrightarrow{\text{as}} A$ and $X \xrightarrow{\text{ms}} B$, then $A \overset{\text{as}}{=} B$.

> *Proof.* For any $\varepsilon > 0$, $\|X[t] - A\| \leq \varepsilon$ and $\|X[t] - B\| \leq \varepsilon$ together imply
>
> $$\|A - B\| = \|A - X[t] + X[t] - B\| \leq \|A - X[t]\| + \|X[t] - B\| \leq 2\varepsilon$$
>
> and so
>
> $$\begin{aligned} \mathsf{P}(\|A - B\| > 2\varepsilon) &\leq \mathsf{P}(\|X[t] - A\| > \varepsilon \text{ or } \|X[t] - B\| > \varepsilon) \\ &\leq \mathsf{P}(\|X[t] - A\| > \varepsilon) + \mathsf{P}(\|X[t] - B\| > \varepsilon) \end{aligned}$$
>
> and the right hand side goes to zero because $X \xrightarrow{\text{p}} A$ and $X \xrightarrow{\text{p}} B$. ■

2.4 Ergodic sequences

In technical applications, the statistical properties of a random process are often inferred by studying a finite number of long (but finite) realisations (i.e. simulations, experiments, or field data). This procedure usually requires the assumption that the process is *ergodic* in some sense. In this section, some ergodicity concepts and theorems are introduced.

2.4.1 Ergodicity in the mean

A constant-mean rs $X[\cdot]$ is said to be (mean square) *ergodic in the mean* if the random sequence

$$M[t] := \frac{1}{t+1} \sum_{k=0}^{t} X[k] \qquad (t \in \mathbb{Z}_+)$$

converges in mean square to μ_X as $t \to \infty$. In this sense, the time average sequence M tells us something about the mean (i.e. the "ensemble average") of the rs X.

Example 2.7. A "random constant" sequence $X[t] = A$ with nondegenerate A is not ergodic in the mean, because $\|M[t] - \mu_X\|^2_{\rm rms} = \|A - \mathsf{E}A\|^2_{\rm rms} = \mathsf{var}A$. ■

We have seen in §2.3.3 that $M \stackrel{\rm ms}{\to} \mu_x$ when $X[\cdot]$ is wss rs with uncorrelated terms. One might then intuitively expect that a wss rs $X[\cdot]$ would be ergodic in the mean if $X[t]$ and $X[t+\tau]$ are sufficiently decorrelated for large values of the time shift τ. This idea is made precise in the following statements.

A necessary and sufficient condition for ergodicity in the mean of a wss random sequence $X[\cdot]$ is

$$\lim_{t\to\infty} \frac{1}{t+1} \sum_{k=0}^{t} \mathrm{tr}\,\mathbf{c}_X[k] = 0 \tag{4}$$

Condition (4) holds if

$$\lim_{t\to\infty} \mathrm{tr}\,\mathbf{c}_X[t] = 0 \tag{5}$$

Condition (5) holds if $\sum_{k=0}^{t} \mathrm{tr}\,\mathbf{c}_X[k]$ converges, in which case

$$\|M[t] - \mu_X\|^2_{\rm rms} \lesssim \frac{1}{t+1} \sum_{k=-\infty}^{\infty} \mathrm{tr}\,\mathbf{c}_X[k]$$

that is, the order of convergence of the rms error is $O(\frac{1}{\sqrt{t}})$. If $\mathrm{tr}\,\mathbf{c}_X[t]$ converges to a nonzero value then the wss rs $X[\cdot]$ is not ergodic in the mean.

Proof: We have

$$\begin{aligned}
\|M[t] - \mu_X\|^2_{\rm rms} &= \mathrm{tr}\,\mathsf{E}(M[t] - \mu_X)(M[t] - \mu_X)' \\
&= \mathrm{tr}\,\mathsf{E}\left((\frac{1}{t+1}\sum_{j=0}^{t} X[j]) - \mu_X \right) \left((\frac{1}{t+1}\sum_{l=0}^{t} X[l]) - \mu_X \right)' \\
&= \frac{1}{(t+1)^2}\,\mathrm{tr} \sum_{j=0}^{t}\sum_{l=0}^{t} \mathbf{c}_X[j,l] = \frac{1}{(t+1)^2} \sum_{j=0}^{t}\sum_{l=0}^{t} \mathrm{tr}\,\mathbf{c}_X[j,l] \\
&= \frac{1}{(t+1)^2} \sum_{k=-t}^{t} (t+1-|k|)\,\mathrm{tr}\,\mathbf{c}_X[k] \\
&= \frac{1}{t+1} \sum_{k=-t}^{t} \left(1 - \frac{|k|}{t+1}\right) \mathrm{tr}\,\mathbf{c}_X[k] \\
&\leq \frac{1}{t+1} \sum_{k=-t}^{t} \mathrm{tr}\,\mathbf{c}_X[k] = \frac{-\mathrm{tr}\,\mathbf{c}_X[0]}{t+1} + \frac{2}{t+1} \sum_{k=0}^{t} \mathrm{tr}\,\mathbf{c}_X[k]
\end{aligned}$$

Thus, a sufficient condition for $\|M[t] - \mu_X\|^2_{\rm rms} \to 0$ is $\frac{1}{t+1}\sum_{k=0}^{t} \mathrm{tr}\,\mathbf{c}_X[k] \to$

0. This is also a necessary condition, because $M \xrightarrow{\text{ms}} \mu_x$ implies that

$$\begin{aligned}\left|\frac{1}{t+1}\sum_{k=0}^{t}\operatorname{tr}\mathbf{c}_X[k]\right| &= \left|\frac{1}{t+1}\sum_{k=0}^{t}\operatorname{tr}\mathsf{E}\big((X[k]-\mu_X)(X[0]-\mu_X)'\big)\right| \\ &= \left|\operatorname{tr}\mathsf{E}\big((M[t]-\mu_X)(X[0]-\mu_X)'\big)\right| \\ &= \left|\mathsf{E}(M[t]-\mu_X)'(X[0]-\mu_X)\right| \\ &\le \underbrace{\|M[t]-\mu_X\|_{\text{rms}}}_{\to 0}\cdot\|X[0]-\mu_X\|_{\text{rms}}\end{aligned}$$

If $\operatorname{tr}\mathbf{c}_X[t] \to \gamma$ then for any given $\varepsilon > 0$ there is some t_0 such that $|\operatorname{tr}\mathbf{c}_X[k]-\gamma| < \varepsilon$ for all $k > t_0$. For any $t > t_0$ we have

$$\begin{aligned}&\left|\left(\frac{1}{t+1}\sum_{k=0}^{t}\operatorname{tr}\mathbf{c}_X[k]\right)-\gamma\right| = \left|\frac{1}{t+1}\sum_{k=0}^{t}(\operatorname{tr}\mathbf{c}_X[k]-\gamma)\right| \\ &\le \frac{1}{t+1}\sum_{k=0}^{t_0}|\operatorname{tr}\mathbf{c}_X[k]-\gamma| + \frac{1}{t+1}\sum_{k=t_0+1}^{t}|\operatorname{tr}\mathbf{c}_X[k]-\gamma| \\ &\le \left(\frac{1}{t+1}\sum_{k=0}^{t_0}(\underbrace{|\operatorname{tr}\mathbf{c}_X[k]|}_{\le \operatorname{tr}\mathbf{c}_X[0]}+|\gamma|)\right)+\varepsilon \le \frac{(t_0+1)(\operatorname{tr}\mathbf{c}_X[0]+|\gamma|)}{t+1}+\varepsilon\end{aligned}$$

and for some $t_1 > t_0$ this is $\le 2\varepsilon$ for all $t > t_1$. Thus, $\frac{1}{t+1}\sum_{k=0}^{t}\operatorname{tr}\mathbf{c}_X[k] \to \gamma$. ■

Example 2.8. Let the random sequences $U[t] \overset{\text{iid}}{\sim} \text{Bernoulli}(\frac{3}{4})$ and $V[t] \overset{\text{iid}}{\sim} \text{Bernoulli}(\frac{1}{4})$ be independent, and let

$$X[t] = A\cdot U[t] + (1-A)\cdot V[t]$$

with $A \sim \text{Bernoulli}(\frac{1}{2})$ independent of U and V. Thus, the outcome of an experiment is that the sample path of the rs X either coincides with the sample path of U or coincides with the sample path of V, with equal probability.

We have

$$\mathsf{E}X[t] = \mathsf{E}A\,\mathsf{E}U[t] + \mathsf{E}(1-A)\,\mathsf{E}V[t] = \tfrac{1}{2}$$

and

$$\begin{aligned}\mathsf{E}X[t+\tau]X[t] &= \mathsf{E}\Big(\big(A\cdot U[t+\tau]+(1-A)\cdot V[t+\tau]\big)\big(A\cdot U[t]+(1-A)\cdot V[t]\big)\Big) \\ &= \mathsf{E}A^2\,\mathsf{E}U[t+\tau]U[t] + \underbrace{\mathsf{E}A(1-A)}_{=0}(\cdots) + \mathsf{E}(1-A)^2\,\mathsf{E}V[t+\tau]V[t] \\ &= \frac{1}{2}\cdot\left(\frac{9}{16}+\frac{3}{16}\delta(\tau)\right)+\frac{1}{2}\cdot\left(\frac{1}{16}+\frac{3}{16}\delta(\tau)\right) \\ &= \frac{5}{16}+\frac{3}{16}\delta(\tau)\end{aligned}$$

and so X is wss. However

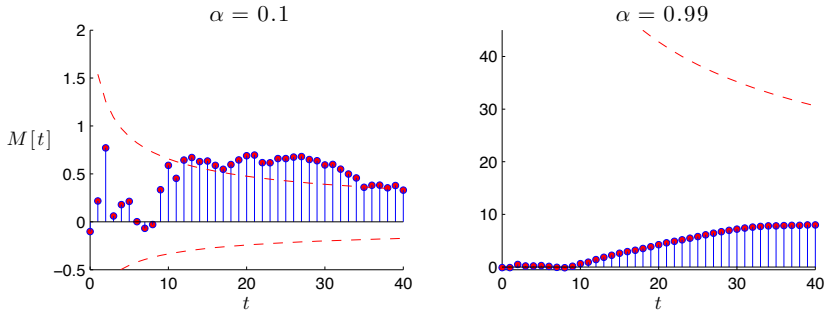
$$c_X[\tau] = \frac{1}{16} + \frac{3}{16}\delta(\tau) \xrightarrow{\tau\to\infty} \frac{1}{16}$$

so X is not ergodic in the mean. ■

Example 2.9. The wss autoregressive sequence presented in §2.2.5 has $c_Y[\tau] = \frac{\alpha^{|\tau|}\sigma^2}{1-\alpha^2}$ and is ergodic in the mean because

$$\sum_{k=-t}^{t} \alpha^{|k|} = 1 + \frac{2\alpha(1-\alpha^t)}{1-\alpha} \xrightarrow[t\to\infty]{} \frac{1+\alpha}{1-\alpha}$$

Here are the sequences of time-averages $M[t] = \frac{1}{t+1}\sum_{k=0}^{t} Y[k]$ for the sample paths that were shown in §2.2.5.



The dashed line in the figure is the "tube" that contains 95% of the sample paths of M, that is, $\mathsf{P}(|M[t]-\mu_Y| \le \rho[t]) = 0.95$; it computed using the approximation

$$\rho[t] := 1.96\|M[t]-\mu_Y\|_{\text{rms}} \lesssim \frac{1.96\sigma}{\sqrt{t+1}(1-\alpha)} \qquad \blacksquare$$

2.4.2 Ergodicity in the correlation

A wss rs $X[\cdot]$ is said to be (mean square) *ergodic in the* i,j,τ *correlation* if the sequence of time averages

$$M_{ij\tau}[t] := \frac{1}{t+1}\sum_{k=0}^{t} X_i[k+\tau]X_j[k] \qquad (t \in \mathbb{Z}_+)$$

converges in mean square to $\mathbf{r}_{Xij}[\tau]$ as $t \to \infty$.

A normal stationary zero-mean rs $X[\cdot]$ is ergodic in the i,j,τ correlation if

$$\sum_{k=0}^{t} \mathbf{r}_{Xii}[k]\mathbf{r}_{Xjj}[k] + \mathbf{r}_{Xij}[k+\tau]\mathbf{r}_{Xij}[k-\tau] \tag{6}$$

converges as $t \to \infty$, and when (6) converges, the order of convergence of $\|M_{ij\tau}[t] - \mathbf{r}_{Xij}[\tau]\|_{\text{rms}}$ is $O(\frac{1}{\sqrt{t}})$. A sufficient condition for convergence of (6) for any i,j,τ is convergence of $\sum_{k=0}^{t} \max_{i,j} |\mathbf{r}_{Xij}[k]|$.

Proof. Let $Z_{ij\tau}[t] = X_i[t+\tau]X_j[t]$, with fixed i, j, τ. Then $Z_{ij\tau}[\cdot]$ is wss, because $\mathsf{E}Z_{ij\tau}[t] = \mathbf{r}_{Xij}[\tau]$ is constant with respect to t and

$$\begin{aligned}\mathsf{E}(Z_{ij\tau}[t_1]Z_{ij\tau}[t_2]) &= \mathsf{E}(X_i[t_1+\tau]X_j[t_1]X_i[t_2+\tau]X_j[t_2]) \\ &= \mathsf{E}(X_i[t_1-t_2+\tau]X_j[t_1-t_2]X_i[\tau]X_j[0])\end{aligned}$$

depends only on the difference $t_1 - t_2$. Using a property of normal rv's[5], we have

$$\begin{aligned}c_{Z_{ij\tau}}[t] &= \mathsf{E}(X_i[t+\tau]X_j[t]X_i[\tau]X_j[0]) - (\mathbf{r}_{Xij}[\tau])^2 \\ &= \mathbf{r}_{Xii}[t]\mathbf{r}_{Xjj}[t] + \mathbf{r}_{Xij}[t+\tau]\mathbf{r}_{Xij}[t-\tau] \qquad (7)\end{aligned}$$

Using the theory presented in the previous section, convergence of $\sum_{k=0}^{t} c_{Z_{ij\tau}}[k]$ is sufficient for $M_{ij\tau} \xrightarrow{\text{ms}} \mathbf{r}_{Xij}[\tau]$ with order of convergence $O(\frac{1}{\sqrt{t}})$.

Suppose $\sum_{k=0}^{t} \max_{i,j} |\mathbf{r}_{Xij}[k]|$ converges, and consider the sum

$$\sum_{k=0}^{t} |c_{Z_{ij\tau}}[k]| \le \sum_{k=0}^{t} |\mathbf{r}_{Xii}[k]\mathbf{r}_{Xjj}[k]| + \sum_{k=0}^{t} |\mathbf{r}_{Xij}[k+\tau]\mathbf{r}_{Xij}[k-\tau]| \quad (8)$$

For the first term in the rhs of (8) we have

$$\sum_{k=0}^{t} |\mathbf{r}_{Xii}[k]\mathbf{r}_{Xjj}[k]| \le \mathbf{r}_{Xjj}[0] \sum_{k=0}^{t} |\mathbf{r}_{Xii}[k]| \le \mathbf{r}_{Xjj}[0] \sum_{k=0}^{t} \max_{i,j} |\mathbf{r}_{Xij}[k]|$$

For the second term in the rhs of (8), choosing t_1 so that $|\mathbf{r}_{Xij}[k-\tau]| \le 1$ for all $k > t_1$, we have

$$\sum_{k=0}^{t} |\mathbf{r}_{Xij}[t+\tau]\mathbf{r}_{Xij}[t-\tau]| \le \sum_{k=0}^{t_1} |\mathbf{r}_{Xij}[t+\tau]\mathbf{r}_{Xij}[t-\tau]| + \underbrace{\sum_{k=t_1+1}^{t} |\mathbf{r}_{Xij}[k+\tau]|}_{\le \sum_{k=t_1+1+\tau}^{t+\tau} \max_{i,j} |\mathbf{r}_{Xij}[k]|}$$

for all $t > t_1$. Thus, convergence of $\sum_{k=0}^{t} \max_{i,j} |\mathbf{r}_{Xij}[k]|$ implies the convergence of $\sum_{k=0}^{t} |c_{Z_{ij\tau}}[k]|$, and thus of $\sum_{k=0}^{t} c_{Z_{ij\tau}}[k]$. ■

Example 2.10. Consider a stationary normal zero-mean autoregressive sequence $X[\cdot]$ with covariance $c_X[\tau] = \alpha^{|\tau|}$, where $|\alpha| < 1$. This rs is ergodic in the correlation for any τ because $\sum_{k=0}^{t} |\alpha^k|$ converges.

[5] If $X_{1:4} \sim \text{Normal}(0, \mathbf{q})$ then $\mathsf{E}X_1X_2X_3X_4 = q_{12}q_{34} + q_{13}q_{24} + q_{14}q_{23}$

For $\tau \geq 0$, we have

$$\begin{aligned}
\sum_{k=-t}^{t} c_{Z_\tau}[k] &= \sum_{k=-t}^{t} (\alpha^{2|k|} + \alpha^{|k+\tau|+|k-\tau|}) \\
&= 1 + 2\sum_{k=1}^{t} \alpha^{2k} + \sum_{k=-\tau}^{\tau} \alpha^{2\tau} + 2\sum_{k=\tau+1}^{t} \alpha^{2k} \\
&\to 1 + \frac{2\alpha^2}{1-\alpha^2} + (2\tau+1)\alpha^{2\tau} + \frac{2\alpha^{2\tau+2}}{1-\alpha^2} \\
&= 2\tau\alpha^{2\tau} + \frac{(1+\alpha^2)(1+\alpha^{2\tau})}{1-\alpha^2}
\end{aligned}$$

and so we have the approximate error bound

$$\|M_\tau[t] - r_X[\tau]\|_{\text{rms}}^2 \lesssim \frac{1}{t+1}\left(2\tau\alpha^{2\tau} + \frac{(1+\alpha^2)(1+\alpha^{2\tau})}{1-\alpha^2}\right) \qquad \blacksquare$$

Example 2.11 Consider the wss rs

$$X[t] = A\cos(\omega t) + B\sin(\omega t)$$

where $\sin\omega \neq 0$ and A and B are nondegenerate uncorrelated zero-mean random variables. The sequence is wss provided that $\mathsf{var}A = \mathsf{var}B$ $(=: \sigma^2)$, and the variance of the rs is then

$$\mathsf{var}X[t] = r_X[0] = \mathsf{E}\big(\frac{A^2+B^2}{2}\big) = \sigma^2$$

The sequence of time averages for estimating this variance is

$$\begin{aligned}
M_0[t] &= \frac{1}{t+1}\sum_{k=0}^{t}\big(A\cos(\omega k) + B\sin(\omega k)\big)^2 \\
&= \tfrac{1}{2}(A^2+B^2) + \sum_{k=0}^{t}\frac{\frac{1}{2}(A^2-B^2)\cos(2\omega k) + AB\sin(2\omega k)}{t+1} \\
&= \tfrac{1}{2}(A^2+B^2) + \tfrac{1}{2}\frac{A^2-B^2}{t+1}\cdot\frac{\sin(\omega(t+1))\cos(\omega t)}{\sin\omega} \\
&\quad + \frac{AB}{t+1}\cdot\frac{\sin(\omega(t+1))\sin(\omega t)}{\sin\omega}
\end{aligned}$$

Then, assuming that A and B have finite fourth moments, we have

$$\begin{aligned}
\|M_0[t] - r_X[0]\|_{\text{rms}}^2 &= \|M_0[t]\|_{\text{rms}}^2 - \sigma^4 \\
&\to \mathsf{E}\left(\big(\frac{A^2+B^2}{2}\big)^2\right) - \left(\mathsf{E}\big(\frac{A^2+B^2}{2}\big)\right)^2 \\
&= \mathsf{var}\big(\frac{A^2+B^2}{2}\big)
\end{aligned}$$

and so the rs X is *not* ergodic in the correlation. It is left as an exercise to show that this rs is ergodic in the mean. $\blacksquare$

3 Power Spectral Density

3.1 Fourier series

3.1.1 Scalar case

Given a real scalar sequence $f[t]$ ($t \in \mathbb{Z}$) that is square summable, i.e. a sequence such that

$$\lim_{t\to\infty} \sum_{k=-t}^{t} f[k]^2 =: \sum_{k=-\infty}^{\infty} f[k]^2$$

exists, the formula

$$\hat{f}(\omega) = \sum_{t=-\infty}^{\infty} f[t] \mathrm{e}^{-\imath\omega t}$$

defines a 2π-periodic function $\hat{f} : \mathbb{R} \to \mathbb{C}$ that is square-integrable on $[-\pi, \pi)$, i.e. $\int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 \,\mathrm{d}\omega < \infty$ and conjugate symmetric, i.e. $\bar{\hat{f}}(\omega) = \hat{f}(-\omega)$. The function $\hat{f}$ is called the *spectrum* of the sequence f.

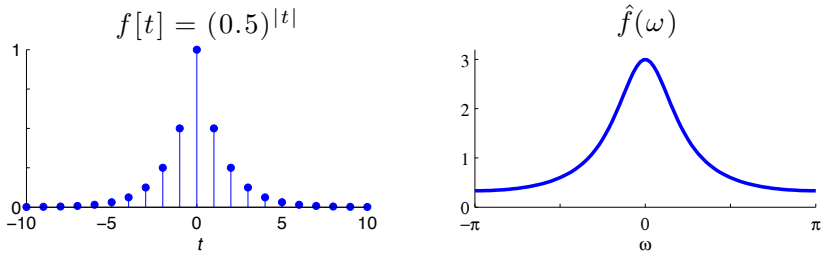
Conversely, given a conjugate-symmetric 2π-periodic function $\hat{f} : [-\pi, \pi) \to \mathbb{C}$ that is square-integrable on $[-\pi, \pi)$, the formula

$$f[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \mathrm{e}^{\imath\omega t} \,\mathrm{d}\omega$$

defines a square-summable real scalar sequence. The sequence terms $f[t]$ are called the *Fourier series coefficients* of $\hat{f}$.

Example 3.1. For $f[t] = \alpha^{|t|}$ with $|\alpha| < 1$ we have

$$\begin{aligned}
\hat{f}(\omega) &= \alpha^0 + \sum_{t=1}^{\infty} \alpha^t \mathrm{e}^{-\imath\omega t} + \sum_{t=1}^{\infty} \alpha^t \mathrm{e}^{\imath\omega t} \\
&= 1 + \sum_{t=1}^{\infty} \left(\alpha \mathrm{e}^{-\imath\omega}\right)^t + \sum_{t=1}^{\infty} \left(\alpha \mathrm{e}^{\imath\omega}\right)^t \\
&= 1 + \frac{\alpha \mathrm{e}^{-\imath\omega}}{1 - \alpha \mathrm{e}^{-\imath\omega}} + \frac{\alpha \mathrm{e}^{\imath\omega}}{1 - \alpha \mathrm{e}^{\imath\omega}} \\
&= \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha\cos(\omega)}
\end{aligned}$$



■

Example 3.2. The Fourier series of a "flat" spectrum $\hat{f}(\omega) = 1$ is

$$f[t] = \delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \blacksquare$$

Linearity:

$$h[t] = \alpha f[t] + \beta g[t] \quad \Leftrightarrow \quad \hat{h}[t] = \alpha \hat{f}[t] + \beta \hat{g}[t]$$

Reversal of a real sequence corresponds to complex conjugation of its spectrum:

$$\overleftarrow{f}[t] = f[-t] \quad \Leftrightarrow \quad \hat{\overleftarrow{f}}(\omega) = \bar{\hat{f}}(\omega)$$

Convolution of sequences corresponds to multiplication of spectra:

$$h = f * g \quad \Leftrightarrow \quad \hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$$

where

$$(f * g)[t] := \sum_{k=-\infty}^{\infty} f[t-k]g[k] = \sum_{k=-\infty}^{\infty} f[k]g[t-k]$$

Taking $t = 0$ we obtain the identity

$$\sum_{k=-\infty}^{\infty} f[k]g[-k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)\hat{g}(\omega)\,\mathrm{d}\omega$$

of which the particular case

$$\sum_{t=-\infty}^{\infty} f[t]^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2\,\mathrm{d}\omega$$

is called *Parseval's identity*.

Termwise multiplication of sequences corresponds to convolution of spectra:

$$b[t] = f[t]g[t] \quad \Leftrightarrow \quad \hat{b} = \hat{f} * \hat{g}$$

where

$$(\hat{f} * \hat{g})(\omega) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega - \lambda)\hat{g}(\lambda)\,\mathrm{d}\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\lambda)\hat{g}(\omega - \lambda)\,\mathrm{d}\lambda$$

Taking $\omega = 0$ we obtain Plancherel's identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\lambda)\bar{\hat{g}}(\lambda)\,\mathrm{d}\lambda = \sum_{t=-\infty}^{\infty} f[t]g[t]$$

of which Parseval's identity is a special case.

3.1.2 Multivariate case

Let $\hat{\mathbf{f}}$ be an $m \times n$ matrix whose elements are conjugate-symmetric 2π-periodic functions that are square-integrable on $[-\pi, \pi)$. Its Fourier coefficients are $m \times n$ real matrices $\mathbf{f}[t]$ whose elements are

$$f_{ij}[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_{ij}(\omega) \mathrm{e}^{\imath \omega t} \, \mathrm{d}\omega$$

and the Fourier series is

$$\hat{\mathbf{f}}(\omega) = \sum_{t=-\infty}^{\infty} \mathbf{f}[t] \mathrm{e}^{-\imath \omega t}$$

Linearity:

$$\mathbf{h}[t] = \alpha \mathbf{f}[t] + \beta \mathbf{g}[t] \quad \Leftrightarrow \quad \hat{\mathbf{h}}[t] = \alpha \hat{\mathbf{f}}[t] + \beta \hat{\mathbf{g}}[t]$$

Two $m \times n$ matrices of conjugate-symmetric spectrum functions are related to their real Fourier series according to:

$$\hat{\mathbf{h}}(\omega) = \hat{\mathbf{f}}(\omega) \hat{\mathbf{g}}'(\omega) \quad \Leftrightarrow \quad \mathbf{h}[t] = \sum_{k=-\infty}^{\infty} \mathbf{f}[k] \mathbf{g}'[k-t] = \sum_{k=-\infty}^{\infty} \mathbf{f}[k+t] \mathbf{g}'[k]$$

where the superscript $'$ denotes the matrix complex conjugate transpose (i.e. Hermitian transpose).

Taking $t = 0$ we obtain Plancherel's identity

$$\sum_{k=-\infty}^{\infty} \mathbf{f}[k] \mathbf{g}'[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mathbf{f}}(\omega) \hat{\mathbf{g}}'(\omega) \, \mathrm{d}\omega$$

Substituting $\mathbf{g} = \mathbf{f}$ and taking the trace of both sides gives Parseval's identity

$$\sum_{k=-\infty}^{\infty} \|\mathbf{f}[k]\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\hat{\mathbf{f}}(\omega)\|^2 \, \mathrm{d}\omega$$

Similarly, the sequence of term-wise products is related to the spectra as follows:

$$\mathbf{h}[t] = \mathbf{f}[t] \, \mathbf{g}'[t] \quad \Leftrightarrow \quad \hat{\mathbf{h}}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mathbf{f}}(\lambda) \hat{\mathbf{g}}'(\lambda - \omega) \, \mathrm{d}\lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mathbf{f}}(\omega + \lambda) \hat{\mathbf{g}}'(\lambda) \, \mathrm{d}\lambda$$

3.2 PSD Scalar Case

3.2.1 Basics

The *power spectral density* (psd) of a scalar wss random sequence $X[\cdot]$ with square-summable autocovariance sequence $c_X[\cdot]$ is the 2π-periodic function $\hat{c}_X$ whose Fourier series coefficients are the autocovariance values, that is,

$$\hat{c}_X(\omega) = \sum_{t=-\infty}^{\infty} c_X[t] \mathrm{e}^{-\imath \omega t}, \quad c_X[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{c}_X(\omega) \mathrm{e}^{\imath \omega t} \, \mathrm{d}\omega$$

Because c_X is real-valued and even, so is $\hat{c}_X$, and the above formulas can be written

$$\hat{c}_X(\omega) = c_X[0] + 2\sum_{t=1}^{\infty} c_X[t]\cos(\omega t), \quad c_X[t] = \frac{1}{\pi}\int_0^{\pi} \hat{c}_X(\omega)\cos(\omega t)\,\mathrm{d}\omega$$

In particular, we have

$$\mathsf{var}X = c_X[0] = \frac{1}{\pi}\int_0^{\pi} \hat{c}_X(\omega)\,\mathrm{d}\omega$$

Example 3.2 (continued). Recall that scalar white noise is a zero-mean wss rs having an autocovariance sequence $c_X[t]$ that is zero for $t \neq 0$. White noise is thus characterised as a rs whose power spectral density is constant. The name "white" comes from the analogy with the flat spectrum of white light in physics. ■

If the autocovariance sequence $c_X[\cdot]$ is absolutely summable, that is,

$$\sum_{t=-\infty}^{\infty} |c_X[t]| < \infty$$

then the psd is continuous and nonnegative, that is, $\hat{c}_X(\omega) \geq 0$ for all ω.

Proof. The psd is continuous because the absolute summability of $c_X[\cdot]$ ensures that the series $\sum_t c_X[t]\cos(\omega t)$ converges uniformly in $\omega \in [-\pi, \pi]$.

For given ω, let

$$\hat{a}[n] = \frac{1}{n+1}\sum_{t=0}^{n}\sum_{s=0}^{n} c_X[s-t]\mathrm{e}^{-\imath(s-t)\omega}$$

By summing along lines of constant $s-t$ we can rewrite this as

$$\hat{a}[n] = c_X[0] + 2\sum_{k=1}^{n}\big(1 - \frac{k}{n+1}\big)c_X[k]\cos(\omega k)$$

For any $1 \leq n_0 < n$ we have

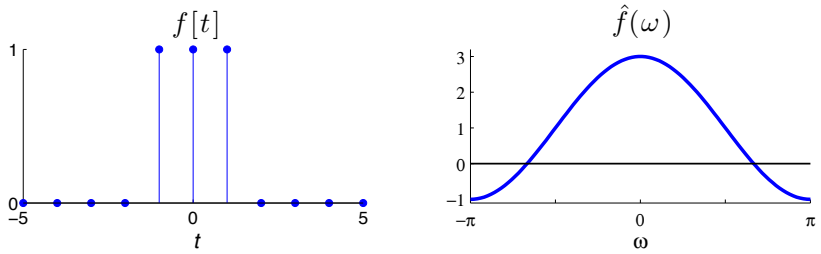
$$\begin{aligned} |\hat{c}_X(\omega) - \hat{a}[n]| &\leq 2\sum_{k=1}^{n_0}\frac{k|c_X[k]|}{n+1} + 2\sum_{k=n_0+1}^{n}\frac{k|c_X[k]|}{n+1} + 2\sum_{k=n+1}^{\infty}|c_X[k]| \\ &\leq \frac{2}{n+1}\sum_{k=1}^{n_0} k|c_X[k]| + 2\sum_{k=n_0+1}^{\infty}|c_X[k]| \end{aligned}$$

Now n_0 can be chosen large enough to make the second term $< \varepsilon$, and then $n_1 > n_0$ can be chosen large enough that the first term is $< \varepsilon$ for all $n > n_1$. Thus $\lim_{n\to\infty} \hat{a}[n] = \hat{c}(\omega)$, and since $\hat{a}[n]$ is nonnegative (because the autocovariance sequence $c_X[\cdot]$ is a nonnegative definite sequence), so is $\hat{c}(\omega)$. ■

Example 3.3. Is the sequence

$$f[t] = 1_{\{-1,0,1\}}(t)$$

a valid autocovariance sequence? It is clearly symmetric, and $|f[t]| \leq f[0]$. However, a plot shows that $\hat{f}(\omega) \geq 0$ does not hold for all ω:



In particular, $\hat{f}(\pi) = 1 + 2\cos(\pi) = -1 < 0$. Thus, $f[\cdot]$ is not nonnegative definite, and so is not a valid autocovariance sequence. ■

Conversely, if $\hat{c}_X : [-\pi, \pi) \to \mathbb{R}$ is a nonnegative even function such that

$$\int_{-\pi}^{\pi} \hat{c}_X(\omega)\, \mathrm{d}\omega < \infty$$

then its Fourier coefficient sequence $c_X[\cdot]$ is a valid autocovariance sequence, that is, $c_X[\cdot]$ is real, even, and nonnegative.

Proof. For any integers $t_1, \ldots, t_m$ and real values $a[1], \ldots, a[m]$ we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a[i]a[j]c_X[t_j - t_i] = \frac{1}{2\pi}\sum_{i=1}^{m}\sum_{j=1}^{m} a[i]a[j]\int_{-\pi}^{\pi} \hat{c}_X(\omega)\mathrm{e}^{\imath\omega(t_j-t_i)}\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{c}_X(\omega)\underbrace{\left(\sum_{i=1}^{m} a[i]\mathrm{e}^{-\imath\omega t_i}\right)\left(\sum_{j=1}^{m} a[j]\mathrm{e}^{\imath\omega t_j}\right)}_{\left|\sum_{j=1}^{m} a[j]\mathrm{e}^{\imath\omega t_j}\right|^2}\mathrm{d}\omega \geq 0$$

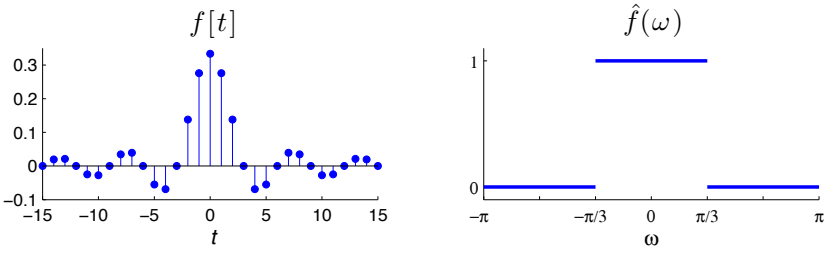
and the proof is complete. ■

Example 3.4. Consider the psd

$$\hat{c}(\omega) = 1_{(-\frac{\pi}{3},\frac{\pi}{3})}(\omega)$$

The corresponding autocovariance sequence is

$$c[t] = \frac{1}{\pi}\int_0^{\pi/3} \cos(\omega t)\,\mathrm{d}\omega = \begin{cases} 1/3 & \text{when } t = 0 \\ \frac{\sin(\pi t/3)}{\pi t} & \text{when } t \neq 0 \end{cases}$$



■

3.2.2 Why it's called "power spectral density"

Let $X[t]$ $(t \in \mathbb{Z})$ be a scalar zero-mean wss random sequence. Let $Y[\cdot]$ be an approximation of X that is constructed by extracting a length-m segment of X and periodically repeating it, that is,

$$Y[0{:}m-1] = X[0{:}m-1], \quad Y[t+km] = Y[t] \quad (t,k \in \mathbb{Z})$$

Its discrete Fourier transform (DFT) is

$$\hat{Y}[p] = \frac{1}{m}\sum_{t=0}^{m-1} Y[t]\mathrm{e}^{-2\pi\imath pt/m}$$

Note that the real and imaginary parts of $\hat{Y}[t]$ $(t \in \mathbb{Z})$ are real random sequences. Also, the sequence $\hat{Y}[\cdot]$ is m-periodic, and because Y is real-valued,

$$\hat{Y}[p] = \bar{\hat{Y}}[-p]$$

The inverse DFT (assuming henceforth that m is even) is

$$\begin{aligned}
Y[t] &= \sum_{p=0}^{m-1} \hat{Y}[p]\mathrm{e}^{2\pi\imath pt/m} \\
&= \hat{Y}[0] + \sum_{p=1}^{\frac{m}{2}-1} \hat{Y}[p]\mathrm{e}^{2\pi\imath pt/m} + \hat{Y}[\tfrac{m}{2}]\mathrm{e}^{-\imath\pi t} + \underbrace{\sum_{p=\frac{m}{2}+1}^{m-1} \hat{Y}[p]\mathrm{e}^{2\pi\imath pt/m}}_{\sum_{q=1}^{\frac{m}{2}-1} \hat{Y}[m-q]\mathrm{e}^{-2\pi\imath qt/m}} \\
&= \hat{Y}[0] + \hat{Y}[\tfrac{m}{2}]\mathrm{e}^{-\imath\pi t} + 2\sum_{p=1}^{\frac{m}{2}-1} \mathrm{Re}\left(\hat{Y}[p]\mathrm{e}^{2\pi\imath pt/m}\right) \\
&= \hat{Y}[0] + \hat{Y}[\tfrac{m}{2}](-1)^t + 2\sum_{p=1}^{\frac{m}{2}-1} \mathrm{Re}(\hat{Y}[p])\cos(\omega_p t) - \mathrm{Im}(\hat{Y}[p])\sin(\omega_p t)
\end{aligned}$$

where $\omega_p = 2\pi p/m$. This can be interpreted as a decomposition of the periodic sequence Y into sinusoidal components. The discrete frequency points $\omega_0, \ldots, \omega_{\frac{m}{2}-1}$ are equally spaced in the interval $[0,\pi)$ with spacing $2\pi/m$.

The random variable

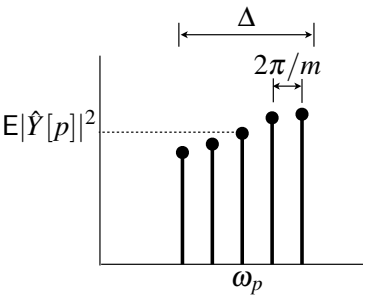
$$P = \frac{1}{m}\sum_{t=0}^{m-1} Y[t]^2$$

can be interpreted as the sum of the "energy" terms $Y[t]^2$ over one period, divided by the length of the period, and is called the (time-)average power of Y. By the Parseval identity for DFT (whose proof is left as an exercise), the average power can be decomposed into the sum of the squared amplitudes of the sinusoidal components:

$$\begin{aligned} P &= \sum_{p=0}^{m-1} |\hat{Y}[p]|^2 = |\hat{Y}[0]|^2 + \sum_{p=1}^{\frac{m}{2}-1} |\hat{Y}[p]|^2 + |\hat{Y}[\frac{m}{2}]|^2 + \sum_{p=\frac{m}{2}+1}^{m-1} |\hat{Y}[p]|^2 \\ &= |\hat{Y}[0]|^2 + |\hat{Y}[\frac{m}{2}]|^2 + 2\sum_{p=1}^{\frac{m}{2}-1} |\hat{Y}[p]|^2 \end{aligned}$$

We see that $2|\hat{Y}[p]|^2$ is the contribution to the average power of Y of the sinusoidal component at the two discrete frequencies $\pm\omega_p$. For convenience, we divide this contribution equally between these two frequencies. The mean (i.e. expected value of) average power at frequency ω_p is then $\mathsf{E}|\hat{Y}[p]|^2$.

Consider a small frequency interval centred at ω_p and of width Δ. The mean average power in the interval is approximately $\mathsf{E}|\hat{Y}[p]|^2$ multiplied by $\frac{\Delta}{2\pi/m}$, the number of discrete frequency points in the interval.



We shall show that as $m \to \infty$, the mean average power in the frequency interval tends towards

$$\frac{1}{2\pi}\int_{\omega_p-\frac{1}{2}\Delta}^{\omega_p+\frac{1}{2}\Delta} \hat{c}_X(\omega)\,\mathrm{d}\omega$$

It is in this sense, then, that $\hat{c}_X(\omega)$ is the "density" of the mean average power in the neighbourhood of angular frequency ω.

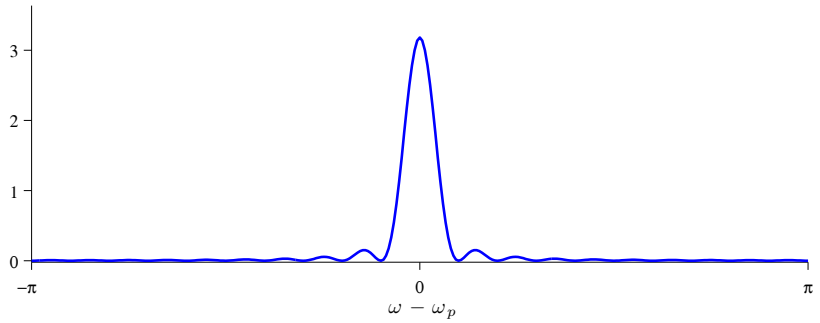
Here we go:

$$\begin{aligned} \frac{m\Delta}{2\pi}\mathsf{E}|\hat{Y}[p]|^2 &= \frac{m\Delta}{2\pi}\mathsf{E}\left(\frac{1}{m}\sum_{t=0}^{m-1} Y[t]\mathrm{e}^{-2\pi\imath pt/m}\right)\cdot\left(\frac{1}{m}\sum_{s=0}^{m-1} Y[s]\mathrm{e}^{2\pi\imath ps/m}\right) \\ &= \frac{\Delta}{2\pi m}\sum_{t=0}^{m-1}\sum_{s=0}^{m-1} c_X[t-s]\mathrm{e}^{-\imath\omega_p(t-s)} \\ &= \frac{\Delta}{2\pi m}\sum_{t=0}^{m-1}\sum_{s=0}^{m-1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{c}_X(\omega)\mathrm{e}^{\imath\omega(t-s)}\,\mathrm{d}\omega\right)\mathrm{e}^{-\imath\omega_p(t-s)} \\ &= \frac{\Delta}{2\pi}\int_{-\pi}^{\pi}\hat{c}_X(\omega)|\hat{h}_m(\omega-\omega_p)|^2\,\mathrm{d}\omega \end{aligned} \tag{9}$$

where

$$
\begin{aligned}
|\hat{h}_m(\omega-\omega_p)|^2 &= \frac{1}{2\pi m}\sum_{t=0}^{m-1}\sum_{s=0}^{m-1} \mathrm{e}^{\imath(\omega-\omega_p)(t-s)} \\
&= \frac{1}{2\pi m}\left(\sum_{t=0}^{m-1} \mathrm{e}^{\imath(\omega-\omega_p)t}\right)\cdot\left(\sum_{s=0}^{m-1} \mathrm{e}^{-\imath(\omega-\omega_p)s}\right) \\
&= \frac{1}{2\pi m}\left(\frac{1-\mathrm{e}^{\imath(\omega-\omega_p)m}}{1-\mathrm{e}^{\imath(\omega-\omega_p)}}\right)\cdot\left(\frac{1-\mathrm{e}^{-\imath(\omega-\omega_p)m}}{1-\mathrm{e}^{-\imath(\omega-\omega_p)}}\right) \\
&= \frac{\sin^2(m(\omega-\omega_p)/2)}{2\pi m \sin^2((\omega-\omega_p)/2)}
\end{aligned}
$$

The function $|\hat{h}_m(\cdot)|^2$ is 2π-periodic. Here's what it looks like for $m = 20$:



The height of the central "lobe" is $\frac{m}{2\pi}$ and its width (distance between the zeros) is $\frac{4\pi}{m}$. For any m, the integral over one period is

$$
\begin{aligned}
\int_{-\pi}^{\pi} |\hat{h}_m(\lambda)|^2 \,\mathrm{d}\lambda &= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi m}\sum_{t=0}^{m-1}\sum_{s=0}^{m-1} \mathrm{e}^{\imath\lambda(t-s)}\right)\mathrm{d}\lambda \\
&= \frac{1}{m}\sum_{t=0}^{m-1}\sum_{s=0}^{m-1} \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\imath\lambda(t-s)}\,\mathrm{d}\lambda}_{1_{\{0\}}(t-s)} \\
&= 1
\end{aligned}
$$

As m increases, the central lobe becomes higher and narrower, the sidelobes become shorter and narrower, and $|\hat{h}_m(\cdot)|^2$ tends toward a "delta" function. As a consequence, the right hand side of (9) approaches $\hat{c}_X(\omega_p)\frac{\Delta}{2\pi}$, which for small Δ is $\approx \frac{1}{2\pi}\int_{\omega_p-\frac{1}{2}\Delta}^{\omega_p+\frac{1}{2}\Delta} \hat{c}_X(\omega)\,\mathrm{d}\omega$. *QED*

3.3 PSD Multivariate Case

3.3.1 Basics

The psd of a vector wss random sequence $X[\cdot]$ is

$$
\hat{\mathbf{c}}_X(\omega) = \sum_{t=-\infty}^{\infty} \mathbf{c}_X[t]\mathrm{e}^{-\imath\omega t}, \quad \mathbf{c}_X[t] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\mathbf{c}}_X(\omega)\mathrm{e}^{\imath\omega t}\,\mathrm{d}\omega
$$

Because $\mathbf{c}_X$ is real-valued and $\mathbf{c}_X[t] = \mathbf{c}_X'[-t]$, it follows that $\hat{\mathbf{c}}_X$ is Hermitian and $\hat{\mathbf{c}}_X(\omega) = \hat{\mathbf{c}}_X^{\top}(-\omega)$ (where the superscript $\top$ denotes the ordinary matrix transpose), and the above formulas can be written

$$\begin{aligned}\hat{\mathbf{c}}_X(\omega) &= \mathbf{c}_X[0] + \sum_{t=1}^{\infty} (\mathbf{c}_X[t] + \mathbf{c}_X'[t]) \cos(\omega t) - \imath \sum_{t=1}^{\infty} (\mathbf{c}_X[t] - \mathbf{c}_X'[t]) \sin(\omega t) \\ \mathbf{c}_X[t] &= \frac{1}{\pi} \int_0^{\pi} \mathrm{Re}(\hat{\mathbf{c}}_X(\omega)) \cos(\omega t)\, \mathrm{d}\omega - \frac{1}{\pi} \int_0^{\pi} \mathrm{Im}(\hat{\mathbf{c}}_X(\omega)) \sin(\omega t)\, \mathrm{d}\omega\end{aligned}$$

In particular, we have

$$\mathsf{var} X = \mathbf{c}_X[0] = \frac{1}{\pi} \int_0^{\pi} \mathrm{Re}(\hat{\mathbf{c}}_X(\omega))\, \mathrm{d}\omega$$

Example 3.5. Let $X[\cdot]$ be a wss random sequence of length-2 vectors having autocovariance sequence

$$\mathbf{c}_X[0] = \mathbf{i}, \quad \mathbf{c}_X[1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \mathbf{c}_X'[-1] \quad \mathbf{c}_X[t] = 0 \ (|t| > 1)$$

Its psd is

$$\hat{\mathbf{c}}_X(\omega) = \begin{bmatrix} 1 & \mathrm{e}^{\imath\omega} \\ \mathrm{e}^{-\imath\omega} & 1 \end{bmatrix} \qquad \blacksquare$$

If the elements of $\mathbf{c}_X[\cdot]$ are absolutely summable then the psd is continuous and nonnegative definite, that is, $a'\hat{\mathbf{c}}(\omega)a \geq 0$ for all a and ω.

Conversely, if $\hat{\mathbf{c}}$ is a nonnegative definite Hermitian matrix-valued function on $[-\pi, \pi)$ that satisfies

$$\hat{\mathbf{c}}(\omega) = \hat{\mathbf{c}}^{\top}(-\omega), \quad \int_{-\pi}^{\pi} \mathrm{tr}\, \hat{\mathbf{c}}(\omega)\, \mathrm{d}\omega < \infty$$

then its Fourier coefficient sequence $\mathbf{c}[\cdot]$ is a valid autocovariance sequence, that is, $\mathbf{c}[\cdot]$ is real, nonnegative definite, and $\mathbf{c}[t] = \mathbf{c}'[-t]$.

3.3.2 Cross-PSD

Consider a vector-valued wss random sequence $Z[\cdot]$ that is partitioned as

$$Z[t] = \begin{bmatrix} X[t] \\ Y[t] \end{bmatrix}$$

The corresponding partitioning of the psd is

$$\hat{\mathbf{c}}_Z(\omega) = \begin{bmatrix} \hat{\mathbf{c}}_X(\omega) & \hat{\mathbf{c}}_{XY}(\omega) \\ \hat{\mathbf{c}}_{YX}(\omega) & \hat{\mathbf{c}}_Y(\omega) \end{bmatrix}$$

The off-diagonal term

$$\hat{\mathbf{c}}_{XY}(\omega) = \sum_{t=-\infty}^{\infty} \mathbf{c}_{XY}[t] \mathrm{e}^{-\imath\omega t}, \quad \mathbf{c}_{XY}[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\mathbf{c}}_{XY}(\omega) \mathrm{e}^{\imath\omega t}\, \mathrm{d}\omega$$

is called the *cross-power spectral density* of X and Y. From the symmetry properties of $\hat{\mathbf{c}}_Z$ we have

$$\hat{\mathbf{c}}_{XY}(\omega) = \hat{\mathbf{c}}'_{YX}(\omega) = \hat{\mathbf{c}}_{YX}{}^{\top}(-\omega)$$

When X and Y are random sequences of equal-length rv's, the cross-psd is related to the psd's by the inequality

$$|\operatorname{tr}\hat{\mathbf{c}}_{XY}(\omega)| \le \sqrt{\operatorname{tr}\hat{\mathbf{c}}_X(\omega)} \cdot \sqrt{\operatorname{tr}\hat{\mathbf{c}}_Y(\omega)}$$

Proof. The 2×2 matrix

$$\begin{bmatrix} \operatorname{tr}\hat{\mathbf{c}}_X(\omega) & \operatorname{tr}\hat{\mathbf{c}}_{XY}(\omega) \\ \operatorname{tr}\hat{\mathbf{c}}_{YX}(\omega) & \operatorname{tr}\hat{\mathbf{c}}_Y(\omega) \end{bmatrix} = \sum_i \begin{bmatrix} \hat{\mathbf{c}}_{X_i}(\omega) & \hat{\mathbf{c}}_{X_iY_i}(\omega) \\ \hat{\mathbf{c}}_{Y_iX_i}(\omega) & \hat{\mathbf{c}}_{Y_i}(\omega) \end{bmatrix}$$

is non-negative definite and so its determinant is non-negative:

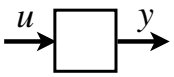
$$\operatorname{tr}\hat{\mathbf{c}}_X(\omega) \cdot \operatorname{tr}\hat{\mathbf{c}}_Y(\omega) - |\operatorname{tr}\hat{\mathbf{c}}_{XY}(\omega)|^2 \ge 0 \qquad \blacksquare$$

4 Linear System with Random Input

4.1 State space model

4.1.1 Basics

A state space model for a linear time-invariant discrete-time system is



$$\begin{aligned} x_{t+1} &= \mathbf{a}x_t + \mathbf{b}u_t \\ y_t &= \mathbf{c}x_t + \mathbf{d}u_t \end{aligned} \tag{10}$$

The times t are integers, the *states* x are n_x vectors, the *input signals* u are n_u vectors, the *output signals* y are n_y vectors, and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are real matrices of conforming dimensions[6].

Given an initial state x_0 and input values $u_0, u_1, \ldots$, the state and output at subsequent times can be found by recursion:

$$x_t = \mathbf{a}^t x_0 + \sum_{k=0}^{t-1} \mathbf{a}^{t-k-1}\mathbf{b}u_k \quad (t \ge 1)$$

$$y_t = \mathbf{c}\mathbf{a}^t x_0 + \mathbf{d}u_t + \sum_{k=0}^{t-1} \mathbf{c}\mathbf{a}^{t-k-1}\mathbf{b}u_k \quad (t \ge 0)$$

[6] To lighten the notation here and in the remainder of these notes, the sequence time index is usually shown as a subscript; the bracket notation will be used only in formulas where the subscripts are showing the indices of vector or matrix components.

Introducing the $n_y \times n_u$ matrix sequence

$$\mathbf{h}_t = \begin{cases} 0, & t < 0 \\ \mathbf{d}, & t = 0 \\ \mathbf{c}\,\mathbf{a}^{t-1}\mathbf{b}, & t \geq 1 \end{cases}$$

the formula for the output can be written as

$$\begin{aligned} y_t &= \mathbf{c}\,\mathbf{a}^t x_0 + \sum_{k=0}^{t} \mathbf{h}_{t-k}\, u_k \\ &= \mathbf{c}\,\mathbf{a}^t x_0 + \sum_{k=0}^{t} \mathbf{h}_k\, u_{t-k} \end{aligned}$$

The matrix sequence $\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \ldots$ is called the *impulse response*, because its jth column is the zero-initial-state output when the input signal is $u_t = \delta(t)\mathbf{i}_{:,j}$, a unit "impulse" in the jth channel. (Here $\mathbf{i}_{:,j}$ denotes the jth column of the identity matrix $\mathbf{i}$.)

The impulse response suffices to completely characterise the system's zero-initial-state output response to any input sequence. In particular, the zero-initial-state output response to a constant unit input in the jth channel is

$$y[t] = \sum_{k=0}^{t} \mathbf{h}_{:,j}[t-k] = \sum_{k=0}^{t} \mathbf{h}_{:,j}[k] \quad (t \geq 0)$$

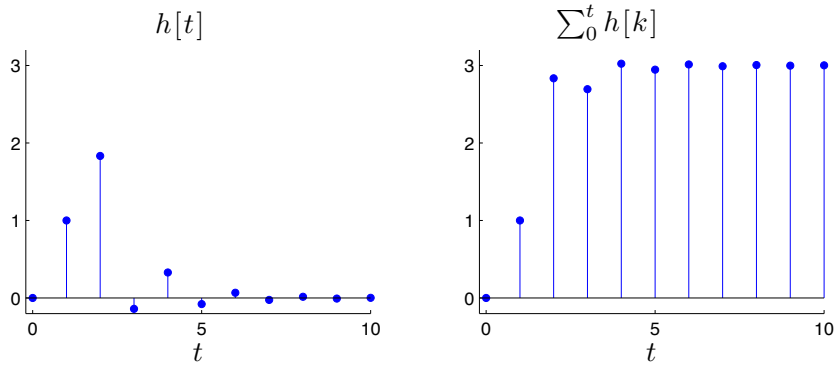
and is called the jth input channel's *step response*.

The system (10) is called *linear* because the output is a linear combination of the initial condition and the inputs. It is called *time-invariant* because the response is invariant to a shift of the time axis origin. The system is *nonanticipative* (or: causal) because the input signals u_{t_2} do not affect the earlier outputs y_{t_1} (where $t_1 < t_2$).

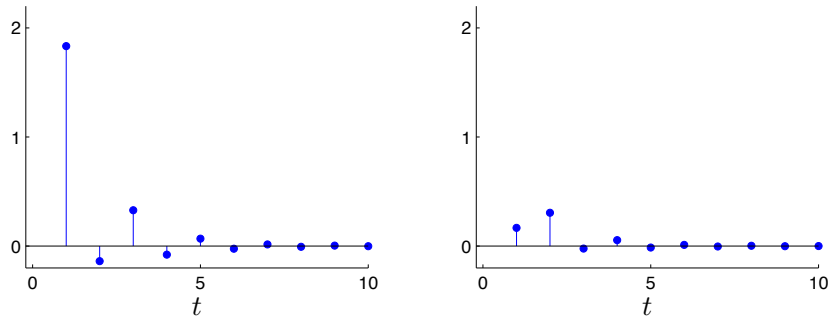
Example 4.1. Consider a single-input single-output system with the state space model matrices

$$\mathbf{a} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \end{bmatrix}$$

The system's impulse response and step response are



The output responses when inputs are zero and initial conditions are $x_0 = \mathbf{i}_{:,1}$ and $x_0 = \mathbf{i}_{:,2}$ are



■

4.1.2 Stability

The linear system is said to be *asymptotically stable* if $\lim_{t\to\infty} x_t = 0$ for any initial state x_0 and zero input.

A necessary and sufficient condition for asymptotic stability is $\rho(\mathbf{a}) < 1$, where the *spectral radius* $\rho(\mathbf{a})$ is the largest eigenvalue modulus of $\mathbf{a}$.

Proof. We use the vector norm $\|z\|_\infty = \max_i |z_i|$ and its subordinate (or: induced) matrix norm

$$\|\mathbf{a}\|_\infty = \max_{x\neq 0} \frac{\|\mathbf{a}\,x\|_\infty}{\|x\|_\infty} = \max_i \sum_j |a_{ij}|$$

To prove sufficiency, suppose $\rho(\mathbf{a}) < 1$. By the Schur factorisation theorem, $\mathbf{a}$ is similar to a (possibly complex) matrix $\mathbf{e}+\mathbf{u}$ with $\mathbf{e}$ a diagonal matrix containing the eigenvalues of $\mathbf{a}$ and $\mathbf{u}$ a strictly upper triangular matrix. Let $\mathbf{s}$ denote the matrix such that $\mathbf{a} = \mathbf{s}^{-1}(\mathbf{e}+\mathbf{u})\mathbf{s}$.

Let $\varepsilon > 0$ and $\mathbf{d}_\varepsilon = \operatorname{diag}(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{n_x})$. We have $\mathbf{d}_\varepsilon^{-1}\,\mathbf{e}\,\mathbf{d}_\varepsilon = \mathbf{e}$ and

$$\mathbf{d}_\varepsilon^{-1}\,\mathbf{u}\,\mathbf{d}_\varepsilon = \begin{bmatrix} 0 & \varepsilon u_{12} & \varepsilon^2 u_{13} & \varepsilon^3 u_{14} & \cdots \\ 0 & 0 & \varepsilon u_{23} & \varepsilon^2 u_{24} & \cdots \\ 0 & 0 & 0 & \varepsilon u_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} =: \tilde{\mathbf{u}}$$

Then, choosing ε small enough that $\|\mathbf{e}\|_\infty + \|\tilde{\mathbf{u}}\|_\infty = \rho(\mathbf{a}) + \|\tilde{\mathbf{u}}\|_\infty$ is smaller than 1, we have

$$\begin{aligned} \|\mathbf{a}^t\|_\infty &= \|\mathbf{s}^{-1}\,\mathbf{d}_\varepsilon\,(\mathbf{e}+\tilde{\mathbf{u}})^t\,\mathbf{d}_\varepsilon^{-1}\,\mathbf{s}\|_\infty \\ &\leq \|\mathbf{s}^{-1}\,\mathbf{d}_\varepsilon\,\|_\infty \big(\|\mathbf{e}\|_\infty + \|\tilde{\mathbf{u}}\|_\infty\big)^t \|\,\mathbf{d}_\varepsilon^{-1}\,\mathbf{s}\|_\infty \xrightarrow{t\to\infty} 0 \end{aligned}$$

and so $\|x_t\|_\infty = \|\mathbf{a}^t\,x_0\|_\infty \leq \|\mathbf{a}^t\|_\infty\,\|x_0\|_\infty \xrightarrow{t\to\infty} 0$.

To prove necessity, suppose $\rho(\mathbf{a}) \geq 1$. Then there exist some λ and v such that $\mathbf{a}v = \lambda v$, $|\lambda| \geq 1$, and $\|v\|_\infty = 1$. It follows that, with $x_0 = v$,

$$\|x_t\|_\infty = \|\mathbf{a}^t x_0\|_\infty = \|\mathbf{a}^t v\|_\infty = \|\lambda^t v\|_\infty = |\lambda^t| \cdot \|v\|_\infty = |\lambda|^t \geq 1$$

and so the system is not asymptotically stable. ■

If $\rho(\mathbf{a}) < 1$ then the (discrete-time) *Lyapunov equation*

$$\mathbf{a}\,\mathbf{x}\,\mathbf{a}' + \mathbf{m} = \mathbf{x}$$

has a unique solution $\mathbf{x}$ for any $n_x \times n_x$ symmetric non-negative definite (snnd) matrix $\mathbf{m}$, and the solution is snnd. If $\rho(\mathbf{a}) < 1$ then the LE has a unique solution $\mathbf{x}$ for any $n_x \times n_x$ symmetric positive definite (spd) matrix $\mathbf{m}$, and the solution is spd. If the LE has a spd solution $\mathbf{x}$ for some spd $\mathbf{m}$ then $\rho(\mathbf{a}) < 1$. If for some spd $\mathbf{m}$ the LE has a symmetric solution $\mathbf{x}$ that is not spd then the system is unstable, that is, there exist some x_0 such that the zero-input system $x_{t+1} = \mathbf{a}x_t$ is unbounded.

Proof. As noted in the previous proof, if $\rho(\mathbf{a}) < 1$ then there exists a nonsingular $\tilde{\mathbf{s}} = \mathbf{d}_\varepsilon^{-1}\mathbf{s}$ such that $\|\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1}\|_\infty < 1$ and $\|(\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1})'\|_\infty < 1$. The fixed-point iteration

$$\mathbf{x}_t = \mathbf{a}\,\mathbf{x}_{t-1}\,\mathbf{a}' + \mathbf{m}$$

has the solution

$$\mathbf{x}_t = \mathbf{a}^t\,\mathbf{x}_0\,(\mathbf{a}')^t + \sum_{k=0}^{t-1} \mathbf{a}^k\,\mathbf{m}\,(\mathbf{a}')^k$$

The sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ converges because

$$\|\,\tilde{\mathbf{s}}\mathbf{a}^t\,\mathbf{x}_0\,(\mathbf{a}')^t\,\tilde{\mathbf{s}}'\,\|_\infty \leq \underbrace{\|\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1}\|_\infty^t}_{\to 0}\ \|\tilde{\mathbf{s}}\mathbf{x}_0\tilde{\mathbf{s}}'\|_\infty\ \underbrace{\|(\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1})'\|_\infty^t}_{\to 0}$$

and

$$\begin{aligned}\|\tilde{\mathbf{s}}\left(\sum_{k=t}^{\infty} \mathbf{a}^k\,\mathbf{m}\,(\mathbf{a}')^k\right)\tilde{\mathbf{s}}'\|_\infty &\leq \sum_{k=t}^{\infty} \|\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1}\|_\infty^k\ \|\tilde{\mathbf{s}}\mathbf{m}\tilde{\mathbf{s}}'\|_\infty\ \|(\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1})'\|_\infty^k \\ &= \underbrace{\|\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1}\|_\infty^t}_{\to 0}\ \underbrace{\|(\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1})'\|_\infty^t}_{\to 0}\ \frac{\|\tilde{\mathbf{s}}\mathbf{m}\tilde{\mathbf{s}}'\|_\infty}{1 - \|\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1}\|_\infty\ \|(\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1})'\|_\infty}\end{aligned}$$

Because the iterands $\mathbf{x}_t$ are snnd, so is $\lim_{t\to\infty} \mathbf{x}_t$.

Furthermore, if $\rho(\mathbf{a}) < 1$ and $\mathbf{x}$, $\mathbf{y}$ are distinct solutions of the LE then

$$\begin{aligned}\|\tilde{\mathbf{s}}(\mathbf{x}-\mathbf{y})\tilde{\mathbf{s}}'\|_\infty &= \|\tilde{\mathbf{s}}\mathbf{a}(\mathbf{x}-\mathbf{y})\mathbf{a}'\tilde{\mathbf{s}}'\|_\infty \\ &\leq \|\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1}\|_\infty\ \|\tilde{\mathbf{s}}(\mathbf{x}-\mathbf{y})\tilde{\mathbf{s}}'\|_\infty\ \|(\tilde{\mathbf{s}}\mathbf{a}\tilde{\mathbf{s}}^{-1})'\|_\infty \\ &< \|\tilde{\mathbf{s}}(\mathbf{x}-\mathbf{y})\tilde{\mathbf{s}}'\|_\infty\end{aligned}$$

which is a contradiction. The case of symmetric positive definite $\mathbf{m}$ is left as an exercise.

If $\mathbf{x}$ is a spd solution of the LE for some spd $\mathbf{m}$, then for any eigenvalue λ of $\mathbf{a}$ with corresponding left eigenvector $y \neq 0$ we have $y'\mathbf{x}y > 0$ and

$$0 < y'\mathbf{m}y = y'(\mathbf{x} - \mathbf{a}\mathbf{x}\mathbf{a}')y = (1 - |\lambda|^2) \cdot y'\mathbf{x}y$$

and so $\rho(\mathbf{a}) < 1$.

The proof of the instability condition is omitted[7]. ■

Example 4.1. (continued) The eigenvalues of $\mathbf{a}$ are $-\frac{1}{2}$ and $\frac{1}{3}$, so $\rho(\mathbf{a}) = \frac{1}{2} < 1$ and the system is asymptotically stable.

The solution of the Lyapunov equation with $\mathbf{m} = \mathbf{i}$ is

$$\mathbf{x} = \frac{1}{168} \begin{bmatrix} 185 & -37 \\ -37 & 353 \end{bmatrix}$$

and its positive definiteness can be shown by exhibiting its Cholesky factorisation

$$\mathbf{x} = \begin{bmatrix} 1.0494 & 0 \\ -0.2099 & 1.4343 \end{bmatrix} \begin{bmatrix} 1.0494 & -0.2099 \\ 0 & 1.4343 \end{bmatrix} \quad ■$$

4.1.3 Transfer Function

The steady-state response of an asymptotically stable system to a sinusoidal input $u_t = \hat{u}\,\mathrm{e}^{\imath\omega t}$ is

$$y_t = \mathbf{c}\mathbf{a}^t x_0 + \sum_{k=0}^{t} \mathbf{h}_k\, \hat{u}\, \mathrm{e}^{\imath\omega(t-k)} \quad \xrightarrow{t\to\infty} \quad \hat{\mathbf{h}}(\omega)\hat{u}\,\mathrm{e}^{\imath\omega t}$$

where $\hat{\mathbf{h}} : \mathbb{R} \to \mathbb{C}^{n_y \times n_u}$ is the spectrum of the impulse response:

$$\hat{\mathbf{h}}(\omega) = \lim_{t\to\infty} \sum_{k=0}^{t} \mathbf{h}_k\, \mathrm{e}^{-\imath\omega k} = \mathbf{d} + \mathbf{c}\left(\sum_{k=1}^{\infty} \mathbf{a}^{k-1} \mathrm{e}^{-\imath\omega k} \right) \mathbf{b} = \mathbf{d} + \mathbf{c}\,(\mathrm{e}^{\imath\omega}\mathbf{i} - \mathbf{a})^{-1}\,\mathbf{b}$$

Defining the system's *transfer function* $\hat{\hat{\mathbf{h}}} : \mathbb{C} \to \mathbb{C}^{n_y \times n_u}$ as

$$\hat{\hat{\mathbf{h}}}(z) = \mathbf{d} + \mathbf{c}\,(z\mathbf{i} - \mathbf{a})^{-1}\,\mathbf{b}$$

we can write $\hat{\mathbf{h}}(\omega) = \hat{\hat{\mathbf{h}}}(\mathrm{e}^{\imath\omega})$. By Cramer's rule we have

$$(z\mathbf{i} - \mathbf{a})^{-1} = \frac{\mathrm{adj}(z\mathbf{i} - \mathbf{a})}{\det(z\mathbf{i} - \mathbf{a})}$$

[7] For a proof see Theorem 4.49 iii in P. J. Antsaklis & A. N. Michel: *A Linear Systems Primer*, Birkhäuser, 2007.

and so each element of the transfer function is a *proper rational function* of z, that is, a ratio of polynomials whose numerator degree is $\leq$ the denominator degree. For an asymptotically stable system, the poles of these rational functions all lie strictly inside the unit circle $|z| = 1$ in the complex plane.

If the input is a constant sequence and the system is asymptotically stable then the output converges to a constant that is given by

$$\lim_{t\to\infty} y_t = \left(\sum_{k=0}^{\infty} \mathbf{h}_k\right) u = \hat{\mathbf{h}}(0)\, u = \hat{\hat{\mathbf{h}}}(1)\, u$$

Example 4.1. (continued) For this system we have

$$(z\mathbf{i} - \mathbf{a})^{-1} = \frac{1}{z(z+\frac{1}{6}) - \frac{1}{6}} \begin{bmatrix} z & \frac{1}{6} \\ 1 & z+\frac{1}{6} \end{bmatrix}$$

and the system's transfer function is

$$\hat{\hat{h}}(z) = \frac{z+2}{z(z+\frac{1}{6}) - \frac{1}{6}} = \frac{z^{-1} + 2z^{-2}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

The value $\hat{\hat{h}}(1) = 3$ corresponds to the limiting value of the system step response that can be seen in the plot shown earlier. ■

4.1.4 ARMA Filter

A scalar autoregressive-moving average (ARMA) filter is described by a difference equation

$$y_t + \alpha_2 y[t-1] + \ldots + \alpha_{n_x+1} y[t-n_x] = \beta_1 u_t + \beta_2 u[t-1] + \ldots + \beta_{n_x+1} u[t-n_x]$$

This can be written as a system of first-order difference equations:

$$\begin{aligned} y[t] &= \beta_1 u[t] + x_1[t] \\ x_1[t+1] &= \beta_2 u[t] + x_2[t] - \alpha_2 y[t] \\ \vdots &= \quad \vdots \\ x_{n_x-1}[t+1] &= \beta_{n_x} u[t] + x_{n_x}[t] - \alpha_{n_x} y[t] \\ x_{n_x}[t+1] &= \beta_{n_x+1} u[t] - \alpha_{n_x+1} y[t] \end{aligned}$$

which corresponds to a state-space model with system matrices[8]

$$\mathbf{a} = \begin{bmatrix} -\alpha_2 & 1 & & & \\ -\alpha_3 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ -\alpha_{n_x} & & & 0 & 1 \\ -\alpha_{n_x+1} & & & & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \beta_2 - \alpha_2\beta_1 \\ \beta_3 - \alpha_3\beta_1 \\ \vdots \\ \beta_{n_x+1} - \alpha_{n_x+1}\beta_1 \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathbf{d} = \beta_1$$

[8] This is the state space model used in Matlab's `filter` and `filtic` functions; Matlab's `tf2ss` function uses a different state space representation.

The characteristic polynomial of $\mathbf{a}$ is

$$\det(\mathbf{i}z - \mathbf{a}) = z^{n_x} + \alpha_2 z^{n_x-1} + \cdots + \alpha_{nx} z + \alpha_{n_x+1}$$

and $\mathbf{a}$ is called the *companion matrix* of this polynomial[9]. The state space system is asymptotically stable iff the zeros of the characteristic polynomial lie strictly inside the unit circle $|z| = 1$.

The ARMA filter's transfer function is

$$\hat{\hat{h}}(z) = \frac{\beta_1 + \beta_2 z^{-1} + \cdots + \beta_{n_x+1} z^{-n_x}}{1 + \alpha_2 z^{-1} + \cdots + \alpha_{n_x+1} z^{-n_x}}$$

4.2 Response to Random Input

4.2.1 Transient Response

Suppose now that the input signal is a random sequence and that the initial state is a random variable. That is, we assume that, in principle, a joint probability distribution for any finite subset of $\{X_0, U_0, U_1, U_2, \ldots\}$ has been specified. One is then interested in characterising the state sequence and output signal sequence

$$\begin{aligned} X_{t+1} &= \mathbf{a}X_t + \mathbf{b}U_t \\ Y_t &= \mathbf{c}X_t + \mathbf{d}U_t \end{aligned} \tag{11}$$

as time marches forward. In this section we focus on formulas for the evolution of the first and second moments.

If the initial state and the inputs are jointly normal (or: Gaussian), then so are the states and the output signals for $t \geq 0$. Then, all affine transformations, marginal rv's or conditional rv's of the random vector

$$\begin{bmatrix} X_{0:t} \\ Y_{0:t} \\ U_{0:t} \end{bmatrix}$$

are normal, because every one of its elements can be obtained as a linear combination of the initial state and of the inputs. Thus, in the Gaussian case, the formulas for first and second moments of signals given in this section provide a complete characterisation of the system's response to random input.

Applying the expectation operator to all the terms in the state space model (11) gives

$$\begin{aligned} \mathsf{E}X_{t+1} &= \mathbf{a}\,\mathsf{E}X_t + \mathbf{b}\,\mathsf{E}U_t \\ \mathsf{E}Y_t &= \mathbf{c}\,\mathsf{E}X_t + \mathbf{d}\,\mathsf{E}U_t \end{aligned}$$

[9] The Matlab function `roots` finds the zeros of a polynomial by computing the eigenvalues of a companion matrix — the opposite of what is done in elementary matrix algebra courses!

The first order moments of the state and output are thus governed by a (deterministic) state space model. Given $\mathsf{E}X_0$ and $\mathsf{E}U_{0,1,2,\ldots}$ we can compute $\mathsf{E}X_{1,2,\ldots}$ and $\mathsf{E}Y_{0,1,2,\ldots}$ by recursion with the state space model, as was done in the case of deterministic signals in §4.1.1.

The signal means being thus fully specified, we can focus our attention on the "centred" random sequences

$$\check{X}_t := X_t - \mathsf{E}X_t, \quad \check{Y}_t := Y_t - \mathsf{E}Y_t, \quad \check{U}_t := U_t - \mathsf{E}U_t$$

These are governed by the state space model

$$\begin{aligned} \check{X}_{t+1} &= \mathbf{a}\check{X}_t + \mathbf{b}\check{U}_t \\ \check{Y}_t &= \mathbf{c}\check{X}_t + \mathbf{d}\check{U}_t \end{aligned}$$

Because $\check{X}_0$ and $\check{U}_{0,1,2,\ldots}$ are zero-mean, so are $\check{X}_{1,2,\ldots}$ and $\check{Y}_{0,1,2,\ldots}$. For the remainder of this section we consider only the centred state, input, and output random sequences, and we omit the inverted-hat (ˇ) over the variable symbol.

Assume that the initial condition is uncorrelated to the inputs, that is, $\mathsf{cov}(X_0, U_t) = 0$ for all $t \geq 0$, and that the input sequence terms are uncorrelated, that is, $\mathsf{cov}(U_{t_1}, U_{t_2}) = 0$ for $t_1 \neq t_2$. First, we note that the state at any epoch is uncorrelated to current and future inputs, that is, $\mathsf{cov}(X[t_1], U[t_2]) = 0$ for $0 \leq t_1 \leq t_2$.

> *Proof.* Induction on t_1. Because the initial condition is uncorrelated to the input signals, we have $\mathsf{cov}(X_0, U_{t_2}) = 0$ for all t_2. Now suppose that for some $t_1 \geq 0$, we have $\mathsf{cov}(X_{t_1}, U_t) = 0$ for all $t \geq t_1$. Then for any $t_2 \geq (t_1 + 1)$ we have
>
> $$\begin{aligned} \mathsf{E}X_{t_1+1}U_{t_2}' &= \mathsf{E}(\mathbf{a}X_{t_1} + \mathbf{b}U_{t_1})U_{t_2}' \\ &= \mathbf{a}\underbrace{\mathsf{cov}(X_{t_1}, U_{t_2})}_{=0} + \mathbf{b}\underbrace{\mathsf{cov}(U_{t_1}, U_{t_2})}_{=0} = 0 \end{aligned}$$ ■

Consequently, for $t \geq 0$ we have

$$\begin{aligned} \mathsf{var}X_{t+1} &= \mathsf{cov}(X_{t+1}, X_{t+1}) = \mathsf{cov}\big(\mathbf{a}X_t + \mathbf{b}U_t, \mathbf{a}X_t + \mathbf{b}U_t\big) \\ &= \mathbf{a}\,\mathsf{cov}(X_t, X_t)\,\mathbf{a}' + \mathbf{a}\,\mathsf{cov}(X_t, U_t)\,\mathbf{b}' + \mathbf{b}\,\mathsf{cov}(U_t, X_t)\,\mathbf{a}' + \mathbf{b}\,\mathsf{cov}(U_t, U_t)\,\mathbf{b}' \\ &= \mathbf{a}\,(\mathsf{var}X_t)\,\mathbf{a}' + \mathbf{b}\,(\mathsf{var}U_t)\,\mathbf{b}' \end{aligned}$$

Thus, given an initial state covariance matrix $\mathsf{var}X_0$ and input covariance matrices $\mathsf{var}U_0, \mathsf{var}U_1, \ldots$, the state covariance matrices $\mathsf{var}X_1, \mathsf{var}X_2, \ldots$, can be found by recursion of the difference equation

$$\mathsf{var}X_{t+1} = \mathbf{a}\,(\mathsf{var}X_t)\,\mathbf{a}' + \mathbf{b}\,(\mathsf{var}U_t)\,\mathbf{b}' \tag{12}$$

This recursion gives the "main diagonal" of the autocovariance (double-indexed) sequence $\mathsf{var}X_t = \mathbf{c}_X[t,t]$. The other elements of the state and output autocovariance sequences can then be computed. For $t + \tau \geq t \geq 0$ we have

$$\begin{aligned} \mathbf{c}_X[t+\tau, t] &= \mathsf{cov}(X_{t+\tau}, X_t) \\ &= \mathsf{cov}\Big(\mathbf{a}^\tau X_t + \sum_{k=0}^{\tau-1} \mathbf{a}^{\tau-k-1}\mathbf{b}U_{t+k},\, X_t\Big) \\ &= \mathbf{a}^\tau\,\mathsf{var}X_t \end{aligned}$$

and

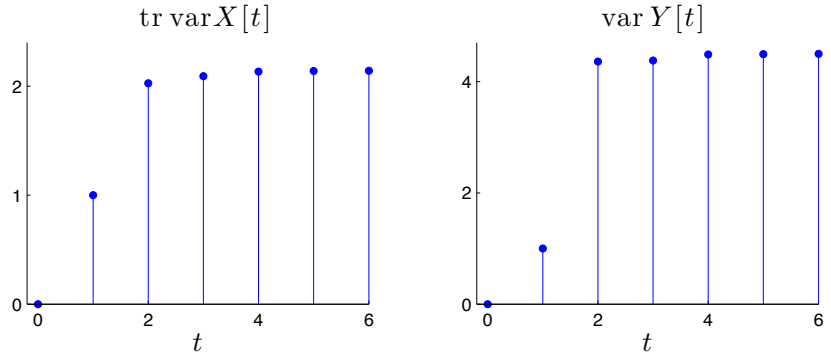
$$\begin{aligned}\mathbf{c}_Y[t+\tau,t] &= \mathsf{cov}(Y_{t+\tau},Y_t)\\ &= \mathsf{cov}\big(\mathbf{c}\,\mathbf{a}^\tau X_t + \sum_{k=0}^{\tau}\mathbf{h}_{\tau-k}U_{t+k},\ \mathbf{c}X_t+\mathbf{d}U_t,\big)\\ &= \mathbf{c}\,\mathbf{c}_X[t+\tau,t]\,\mathbf{c}' + \mathbf{h}_\tau\,\mathsf{var}U_t\,\mathbf{d}'\end{aligned}$$

In particular,

$$\mathsf{var}Y_t = \mathbf{c}\,\mathsf{var}X_t\,\mathbf{c}' + \mathbf{d}\,\mathsf{var}U_t\,\mathbf{d}'$$

Note that, even if the input is wide-sense stationary, the state and output is not generally wide-sense stationary, because of the start-up transients.

Example 4.1. (continued) The variance of the state and output when the initial state is $\mathsf{var}X_0 = 0$ and the input is unit white noise $\mathsf{var}U_t = 1$ evolve as follows.



■

4.2.2 Steady-State Response to White Noise Input

Here we continue to consider centred signals and to assume that the initial state is uncorrelated to the inputs and that the input sequence terms are uncorrelated. We further assume that the system is asymptotically stable and that the input signal is wss (i.e. it is white noise); we denote $\mathsf{var}U_t =: \mathbf{q}$.

With these assumptions, $\mathsf{var}X_t \xrightarrow{t\to\infty} \mathbf{x}$, where $\mathbf{x}$ is the solution of the discrete Lyapunov equation

$$\mathbf{x} = \mathbf{a}\,\mathbf{x}\,\mathbf{a}' + \mathbf{b}\,\mathbf{q}\,\mathbf{b}'$$

Proof. The recursion (12) for $\mathsf{var}X_t$ coincides with the fixed-point iteration in the proof of §4.1.2, with $\mathbf{m} = \mathbf{b}\,\mathbf{q}\,\mathbf{b}'$. This fixed-point iteration was shown to converge to $\mathbf{x}$ when $\rho(\mathbf{a}) < 1$.

Then, for any fixed $\tau \geq 0$ we have

$$\mathbf{c}_X[t+\tau,t] = \mathbf{a}^\tau\,\mathsf{var}X_t \quad \xrightarrow{t\to\infty} \quad \mathbf{a}^\tau\,\mathbf{x}$$

while for $\tau \leq 0$

$$\mathbf{c}_X[t+\tau,t] = \mathbf{c}'_X[t+\tau-\tau,t+\tau] = \mathsf{var}X_{t+\tau}\,(\mathbf{a}')^{-\tau} \quad \xrightarrow{t\to\infty} \quad \mathbf{x}\,(\mathbf{a}')^{-\tau}$$

We see that the second-order moments of the centred state rs $X[\cdot]$ approach those of a zero-mean wss rs $\tilde{X}$ having the autocovariance sequence

$$\mathbf{c}_{\tilde{X}}[t] = \begin{cases} \mathbf{a}^t\,\mathbf{x} & (t \geq 0) \\ \mathbf{x}\,(\mathbf{a}')^{-t} & (t \leq 0) \end{cases}$$

Such a *steady-state response* is obtained as the response to the state evolution recursion

$$\tilde{X}_{t+1} = \mathbf{a}\,\tilde{X}_t + \mathbf{b}\,U_t$$

with $\mathsf{E}\tilde{X}_0 = 0$ and $\mathsf{var}\tilde{X}_0 = \mathbf{x}$.

Under the assumptions stated at the beginning of this section, the centred response X_t converges to the steady-state response $\tilde{X}_t$, in the sense that $X_t - \tilde{X}_t \xrightarrow[t\to\infty]{\text{ms}} 0$.

Proof. Let $E_t = X_t - \tilde{X}_t$. Subtracting the state evolution formulas

$$X_{t+1} = \mathbf{a}\,X_t + \mathbf{b}\,U_t, \quad \tilde{X}_{t+1} = \mathbf{a}\,\tilde{X}_t + \mathbf{b}\,U_t$$

we obtain the recursion $E_{t+1} = \mathbf{a}\,E_t$. The recursion for the covariance matrix sequence is $\mathsf{var}E_{t+1} = \mathbf{a}\,\mathsf{var}E_t\,\mathbf{a}'$. Then, because $\rho(\mathbf{a}) < 1$, we have $\mathsf{var}E_t \xrightarrow{t\to\infty} 0$. ■

The steady-state output response follows the recursion $\tilde{Y}_t = \mathbf{c}\,\tilde{X}_t + \mathbf{d}\,U_t$, and has the autocovariance sequence

$$\mathbf{c}_{\tilde{Y}}[t] = \begin{cases} \mathbf{c}\,\mathbf{c}_{\tilde{X}}[t]\,\mathbf{c}' + \mathbf{h}_t\,\mathbf{q}\,\mathbf{d}' & (t \geq 0) \\ \mathbf{c}\,\mathbf{c}_{\tilde{X}}[t]\,\mathbf{c}' + \mathbf{d}\,\mathbf{q}\,\mathbf{h}'_{-t} & (t \leq 0) \end{cases}$$

In particular,

$$\mathsf{var}\,\tilde{Y} = \mathbf{c}_{\tilde{Y}}[0] = \mathbf{c}\,\mathbf{x}\,\mathbf{c}' + \mathbf{d}\,\mathbf{q}\,\mathbf{d}'$$

If the initial state happens to be $\mathsf{var}\,X_0 = \mathbf{x}$ then the state response is the same as the steady-state response: there is no "start-up transient" and the state rs is "instantly" stationary. For computer simulations, an initial state having variance $\mathbf{x}$ can be generated by multiplying a sample of a zero-mean rv having covariance matrix $\mathbf{i}$, such as Normal$(0,\mathbf{i})$, by a matrix $\mathbf{u}$ that is a "square root" of $\mathbf{x}$ in the sense that $\mathbf{u}\,\mathbf{u}' = \mathbf{x}$. In Matlab or Octave this can be done with the command

```
initial_state = chol(x,'lower') * randn(size(x,1),1)
```

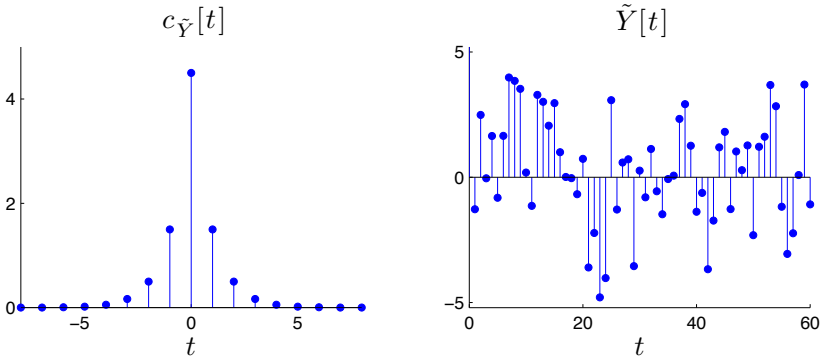
Example 4.1. (continued) With $\mathsf{var}\,U_t = 1$, the solution of the discrete Lyapunov equation (obtained by hand calculation or using `dlyap` in the Matlab Control Toolbox) and its Cholesky factorisation is

$$\mathbf{x} = \begin{bmatrix} 1.0714 & -0.2143 \\ -0.2143 & 1.0714 \end{bmatrix} = \begin{bmatrix} 1.0351 & 0 \\ -0.2070 & 1.0142 \end{bmatrix} \begin{bmatrix} 1.0351 & -0.2070 \\ 0 & 1.0142 \end{bmatrix}$$

and the steady-state output variance is

$$\operatorname{var}\tilde{Y} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1.0714 & -0.2143 \\ -0.2143 & 1.0714 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 = 4.5$$

Here is a plot of the output autocovariance sequence and of an output sample path generated with $X_0 \sim \text{Normal}(0, \mathbf{x})$ and Gaussian unit white noise $U_t \overset{\text{iid}}{\sim} \text{Normal}(0,1)$.



■

The steady-state state's psd is

$$\begin{aligned}
\hat{\mathbf{c}}_{\tilde{X}}(\omega) &= \sum_{t=-\infty}^{\infty} \mathrm{e}^{-\imath\omega t}\mathbf{c}_{\tilde{X}}[t] \\
&= -\mathbf{c}_{\tilde{X}}[0] + \sum_{t=0}^{\infty} \mathrm{e}^{-\imath\omega t}\mathbf{c}_{\tilde{X}}[t] + \sum_{t=0}^{\infty} \mathrm{e}^{\imath\omega t}\mathbf{c}'_{\tilde{X}}[t] \\
&= -\mathbf{x} + \Big(\sum_{t=0}^{\infty} (\mathrm{e}^{-\imath\omega}\mathbf{a})^t\Big)\mathbf{x} + \mathbf{x}\Big(\sum_{t=0}^{\infty} (\mathrm{e}^{\imath\omega}\mathbf{a}')^t\Big) \\
&= -\mathbf{x} + (\mathbf{i} - \mathrm{e}^{-\imath\omega}\mathbf{a})^{-1}\mathbf{x} + \mathbf{x}(\mathbf{i} - \mathrm{e}^{\imath\omega}\mathbf{a}')^{-1} \\
&= (\mathbf{i} - \mathrm{e}^{-\imath\omega}\mathbf{a})^{-1}\Big(-(\mathbf{i} - \mathrm{e}^{-\imath\omega}\mathbf{a})\mathbf{x}(\mathbf{i} - \mathrm{e}^{\imath\omega}\mathbf{a}') \\
&\quad + \mathbf{x}(\mathbf{i} - \mathrm{e}^{\imath\omega}\mathbf{a}') + (\mathbf{i} - \mathrm{e}^{-\imath\omega}\mathbf{a})\mathbf{x}\Big)(\mathbf{i} - \mathrm{e}^{\imath\omega}\mathbf{a}')^{-1} \\
&= (\mathbf{i} - \mathrm{e}^{-\imath\omega}\mathbf{a})^{-1}\mathbf{b}\,\mathbf{q}\,\mathbf{b}'(\mathbf{i} - \mathrm{e}^{\imath\omega}\mathbf{a}')^{-1} \\
&= (\mathrm{e}^{\imath\omega}\mathbf{i} - \mathbf{a})^{-1}\mathbf{b}\,\mathbf{q}\,\mathbf{b}'(\mathrm{e}^{-\imath\omega}\mathbf{i} - \mathbf{a}')^{-1}
\end{aligned}$$

The steady-state output's psd is

$$\begin{aligned}
\hat{\mathbf{c}}_{\tilde{Y}}(\omega) &= \mathbf{c}\,\hat{\mathbf{c}}_{\tilde{X}}(\omega)\,\mathbf{c}' - \mathbf{d}\,\mathbf{q}\,\mathbf{d}' + \Big(\sum_{t=0}^{\infty} \mathbf{h}_t \mathrm{e}^{-\imath\omega t}\Big)\mathbf{q}\,\mathbf{d}' + \mathbf{d}\,\mathbf{q}\Big(\sum_{t=0}^{\infty} \mathbf{h}'_t \mathrm{e}^{\imath\omega t}\Big) \\
&= (\hat{\mathbf{h}}(\omega) - \mathbf{d})\,\mathbf{q}\,(\hat{\mathbf{h}}(\omega) - \mathbf{d})' - \mathbf{d}\,\mathbf{q}\,\mathbf{d}' + \hat{\mathbf{h}}(\omega)\mathbf{q}\,\mathbf{d}' + \mathbf{d}\,\mathbf{q}\,\hat{\mathbf{h}}'(\omega) \\
&= \hat{\mathbf{h}}(\omega)\,\mathbf{q}\,\hat{\mathbf{h}}'(\omega)
\end{aligned}$$

In particular, the output psd of a single-input single-output system is

$$\hat{c}_{\tilde{Y}}(\omega) = |\hat{h}(\omega)|^2 q$$

Example 4.1. (continued) The steady-state state's psd is

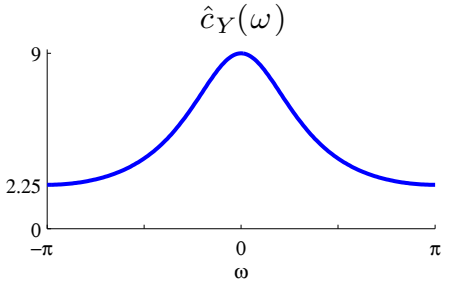
$$\begin{aligned}\hat{\mathbf{c}}_{\tilde{X}}(\omega) &= \frac{1}{|e^{\imath\omega}(e^{\imath\omega}+\frac{1}{6})-\frac{1}{6}|^2}\begin{bmatrix} e^{\imath\omega} & \frac{1}{6} \\ 1 & e^{\imath\omega}+\frac{1}{6}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} e^{-\imath\omega} & 1 \\ \frac{1}{6} & e^{-\imath\omega}+\frac{1}{6}\end{bmatrix} \\ &= \frac{18}{19+5\cos(\omega)-6\cos(2\omega)}\begin{bmatrix}1 & e^{\imath\omega}\\ e^{-\imath\omega} & 1\end{bmatrix}\end{aligned}$$

The steady-state output's psd is

$$\hat{\mathbf{c}}_{\tilde{Y}}(\omega) = \begin{bmatrix}1 & 2\end{bmatrix}\hat{\mathbf{c}}_{\tilde{X}}(\omega)\begin{bmatrix}1\\2\end{bmatrix}+0 = \frac{18(5+4\cos(\omega))}{19+5\cos(\omega)-6\cos(2\omega)}$$

The same formula is found using the transfer function:

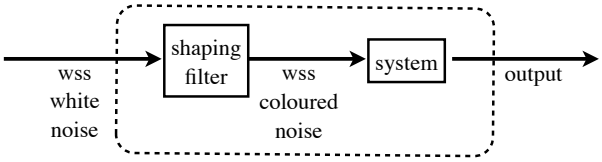
$$|\hat{h}(\omega)|^2 = \left|\frac{e^{\imath\omega}+2}{e^{\imath\omega}(e^{\imath\omega}+\frac{1}{6})-\frac{1}{6}}\right|^2 = \frac{18(5+4\cos(\omega))}{19+5\cos(\omega)-6\cos(2\omega)}$$



4.2.3 Design of a Shaping Filter

In the previous section it was shown how to determine the steady-state response of an asymptotically stable system to a white noise input. Now we consider a more general wss input, one whose psd $\hat{\mathbf{c}}_U$ is not necessarily constant.

Suppose the wss input can be modelled as the white noise response of an asymptotically stable state space system (called a *shaping filter*). The system that is obtained by connecting the shaping filter and the original system in series is called an *augmented system*.



The response of the original system to the coloured noise can then be found using the methods of the previous section: it is the response of the augmented system to a wss white noise.

Let's consider the task of designing a single-input single-output shaping filter. Given a rational scalar psd $\hat{c}(\omega)$, we want to find an asymptotically stable system

whose impulse response spectrum satisfies

$$|\hat{h}(\omega)|^2 = \hat{c}(\omega)$$

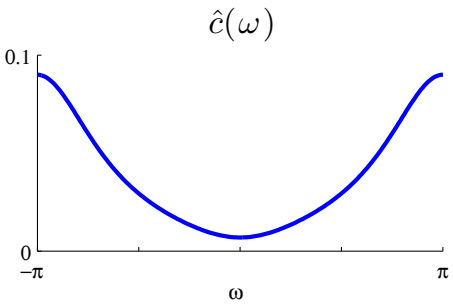
That is, we want to find an asymptotically stable ARMA filter transfer function $\hat{\hat{h}}(z)$ such that

$$\hat{\hat{h}}(z)\hat{\hat{h}}(z^{-1}) = \hat{\hat{c}}(z)$$

where $\hat{\hat{c}}(e^{\imath\omega}) = \hat{c}(\omega)$. This *spectral factorisation* of the psd can be done by factoring the numerator and denominator of $\hat{\hat{c}}$. The process is illustrated by an example, as follows.

Example 4.2. Suppose we want to design a shaping filter for the psd

$$\hat{c}(\omega) = \frac{5 - 4\cos\omega}{146 + 22\cos\omega - 24\cos 2\omega}$$



Denoting $z = e^{\imath\omega}$, the psd denominator is factored as

$$\begin{aligned}
&146 + 22\cos\omega - 24(2\cos^2\omega - 1) \\
&= 170 + 22\cos\omega - 48\cos^2\omega \\
&= -48(\cos\omega + \tfrac{5}{3})(\cos\omega - \tfrac{17}{8}) \\
&= -48(\tfrac{z+z^{-1}}{2} + \tfrac{5}{3})(\tfrac{z+z^{-1}}{2} - \tfrac{17}{8}) \\
&= (z^{-1}+3)(z+3)\cdot(z^{-1}-4)(z-4) \\
&= (z^{-1}+3)(z^{-1}-4)\cdot(z+3)(z-4) \\
&= (z^{-2} - z^{-1} - 12)\cdot(z^2 - z - 12)
\end{aligned}$$

The numerator is factored similarly:

$$-4(\cos\omega - \tfrac{5}{4}) = -4(\tfrac{z+z^{-1}}{2} - \tfrac{5}{4}) = (z^{-1} - 2)\cdot(z - 2)$$

The shaping filter's transfer function is then

$$\hat{\hat{h}}(z) = \frac{z^{-1} - 2}{z^{-2} - z^{-1} - 12} = \frac{2 - z^{-1}}{12 + z^{-1} - z^{-2}}$$

It has poles at $-\frac{1}{3}$ and $\frac{1}{4}$, and a zero at $\frac{1}{2}$. A state space model for this shaping filter can be realised using the formulas given in §4.1.4. ■

There exist many algorithms for computing a spectral factorisation; see

A.H. Sayed & T. Kailath, A survey of spectral factorization methods, *Numerical Linear Algebra with Applications* (2001), vol 8, p. 467–496.

The following Matlab function is an implementation of one of the algorithms presented in that paper.

```
function g=specfac(p)
% Given a Laurent polynomial
%   P(z)=p(1) + p(2)/z + p(2)'*z + ... + p(m+1)/z^m + p(m+1)'*z^m
% which is positive on the unit circle,
%      g=specfac(p)
% finds the coefficients of the factorization
%   P(z)=(g(1)+g(2)/z+ ... +g(m+1)/z^m)*(g(1)'+g(2)'*z+ ... +g(m+1)'*z^m)
% with poles of g(1)+ g(2)/z + ... + g(m+1)/z^m inside unit circle.
%
% Uses: dare (Matlab Control Toolbox)

% author: Robert Piche, Tampere University of Tampere, 2010

m=length(p)-1;
if m==0, g=sqrt(p); return, end
F=diag(ones(1,m-1),-1);
h=[zeros(1,m-1) 1];
N=p(end:-1:2); N=N(:);
S=dare(F',h',zeros(m),-p(1),-N);
Fp=F-((F*S*h'-N)/(h*S*h'-p(1)))*h;
rp=p(1)-h*S*h';
g=(N-F*S*h')/rp;
g=sqrt(rp)*[1;g(end:-1:1)];
```

Example 4.2. (continued) A factorisation of the denominator

$$146+22\cos\omega-24\cos 2\omega = 146+\frac{22}{2}(z+z^{-1})-\frac{24}{2}(z^2+z^{-2})$$

can be found like this:

```
>> specfac([146,22/2,-24/2])

ans =

   12.0000
    1.0000
   -1.0000
```

This agrees with the factorisation found earlier, except for the sign. ■

In spectral factorisation of a rational psd function, the transfer function numerator zeros are conventionally selected so that they lie inside the unit circle. Then the reciprocal of the transfer function is also a stable filter. This is called a *whitening filter*, because when it is fed a wss zero-mean input having psd $\hat{c}(\omega)$, its steady-state output is a unit-variance white noise.

Example 4.2. (continued) A whitening filter is

$$\frac{1}{\hat{\hat{h}}(z)} = \frac{12 + z^{-1} - z^{-2}}{2 - z^{-1}} \qquad \blacksquare$$

5 Filtering

5.1 Formulation

We consider the following linear discrete-time state space model

$$\begin{aligned} X_{t+1} &= \mathbf{a}X_t + \mathbf{b}W_t \quad (t \in \mathbb{Z}_+) \\ Y_t &= \mathbf{c}X_t + V_t \quad (t \in \mathbb{N}) \end{aligned}$$

We assume that $\{X_0,\ W_{0:t},\ V_{1:t}\}$ are jointly normal and uncorrelated, with $X_0 \sim \text{Normal}(\hat{x}_0,\ \mathbf{p}_0)$, $W_t \sim \text{Normal}(0,\ \mathbf{q})$, and $V_t \sim \text{Normal}(0,\ \mathbf{r})$ with nonsingular $\mathbf{r}$.

The filtering problem is to find the probability distribution of the posterior $X_t \,|\, (Y_{1:t} = y_{1:t})$, that is, of the current state given the current and past observations. In section 5.2 we show how the filtering problem is in fact a special case of the estimation problem for random variables that was studied in §1.6, and so this problem's solution formula could, in principle, be applied. However, such a "batch" solution usually requires too much memory and computation to be feasible in practice. In section 5.3 we present a recursive algorithm known as the *Kalman filter* [10] that updates the estimate as each new measurement arrives.

5.2 Batch Solution

The filtering problem's state space model equations can be combined into

$$Y_{1:t} = \mathbf{h}Z + V_{1:t}$$

with

$$\mathbf{h} = \text{diag}(\mathbf{c}, \ldots, \mathbf{c})\,\mathbf{g}, \quad \mathbf{g} = \begin{bmatrix} \mathbf{a} & \mathbf{b} & & & \\ \mathbf{a}^2 & \mathbf{ab} & \mathbf{b} & & \\ \vdots & & & \ddots & \\ \mathbf{a}^t & \mathbf{a}^{t-1}\mathbf{b} & \cdots & \mathbf{ab} & \mathbf{b} \end{bmatrix}, \quad Z = \begin{bmatrix} X_0 \\ W_0 \\ \vdots \\ W_{t-1} \end{bmatrix}$$

[10] In R. Kalman's original paper in 1960, the algorithm was derived using least-squares estimation concepts. A probabilistic derivation of the algorithm was published earlier by R. L. Stratonovich, and rediscovered later (independently) by Y. Ho and R. Lee; see "A Bayesian approach to problems in stochastic estimation and control" in *IEEE Transactions on Automatic Control* vol. 9, pp. 333–339, Oct. 1964.

This is a measurement system model of the form studied in §1.6. Indeed, we have uncorrelated jointly normal rv's Z and $V_{1:t}$, with $Z \sim \text{Normal}(\hat{z}^-,\ \mathbf{p}_z^-)$,

$$\hat{z}^- = \begin{bmatrix} \hat{x}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{p}_z^- = \text{diag}(\mathbf{p}_0, \mathbf{q}, \ldots, \mathbf{q})$$

and $V_{1:t} \sim \text{Normal}(0,\ \text{diag}(\mathbf{r}, \ldots, \mathbf{r}))$. Then, as in §1.6, the posterior distribution is $Z \,|\, (Y_{1:t} = y_{1:t}) \sim \text{Normal}(\hat{z}^+,\ \mathbf{p}_z^+)$ with

$$\begin{aligned} \mathbf{k} &= \mathbf{p}_z^- \mathbf{h}' \left(\mathbf{h} \mathbf{p}_z^- \mathbf{h}' + \text{diag}(\mathbf{r}, \ldots, \mathbf{r}) \right)^{-1} \\ \hat{z}^+ &= \hat{z}^- + \mathbf{k} \left(y_{1:t} - \mathbf{h} \hat{z}^- \right) \\ \mathbf{p}_z^+ &= \mathbf{p}_z^- - \mathbf{k} \mathbf{h} \mathbf{p}_z^- \end{aligned}$$

In particular, because $X_{1:t} = \mathbf{g}Z$, we have the posterior distribution of the entire state history:

$$X_{1:t} \,|\, (Y_{1:t} = y_{1:t}) \sim \text{Normal}(\mathbf{g}\hat{z}^+,\ \mathbf{g}\mathbf{p}_z^+\mathbf{g}')$$

This is the complete solution of the filtering problem, but it is impractical for large t, because the matrices and vectors grow with t. A more practical solution is described in the next section.

5.3 Recursive Solution (Kalman Filter)

Let $\mathscr{Y}_t$ denote the observation history $Y_{1:t} = y_{1:t}$. The recursive filter processes the measurements at each time step in two stages. In the first stage, it takes the previous time step's estimate $X_{t-1} \,|\, \mathscr{Y}_{t-1}$ (which is just X_0 when $t = 1$), and uses the state evolution model to obtain the probability distribution of the *predicted* current state $X_t \,|\, \mathscr{Y}_{t-1}$. It then uses the current measurement $Y_t = y_t$ to *update* this probability distribution and obtain the estimate $X_t \,|\, \mathscr{Y}_t$.

5.3.1 Prediction

The rv $X_{t-1} \,|\, \mathscr{Y}_{t-1}$ is normal, because it is a marginal of $X_{1:t-1} \,|\, \mathscr{Y}_{t-1}$, which from the theory presented in §5.2 is normal. Also, $W_{t-1} = W_{t-1} \,|\, \mathscr{Y}_{t-1}$, because $Y_{1:t-1}$, being a linear combination of $\{X_0,\ W_{0:t-2},\ V_{1:t-1}\}$, is independent of W_{t-1}. Furthermore, $X_{t-1} \,|\, \mathscr{Y}_{t-1}$ and W_{t-1} are jointly normal and independent, because

$$\begin{aligned} p(x_{t-1}, w_{t-1} \,|\, y_{1:t-1}) &= \frac{p(x_{t-1}, w_{t-1}, y_{1:t-1})}{p(y_{1:t-1})} \\ &= \frac{p(x_{t-1}, y_{1:t-1})\, p(w_{t-1})}{p(y_{1:t-1})} \\ &= p(x_{t-1} \,|\, y_{1:t-1}) p(w_{t-1}) \end{aligned}$$

Then the *one-step prediction* $X_t \,|\, \mathscr{Y}_{t-1}$ is normal, because it is a linear combination of $X_{t-1} \,|\, \mathscr{Y}_{t-1}$ and $W_{t-1} \,|\, \mathscr{Y}_{t-1}$.

Let $\hat{x}_{t-1}$ and $\mathbf{p}_{t-1}$ denote the mean and covariance of $X_{t-1} \,|\, \mathscr{Y}_{t-1}$. The mean of the one-step prediction is

$$\begin{aligned} \mathsf{E}(X_t \,|\, \mathscr{Y}_{t-1}) &= \mathsf{E}(\mathbf{a} X_{t-1} + \mathbf{b} W_{t-1} \,|\, \mathscr{Y}_{t-1}) \\ &= \mathbf{a}\, \mathsf{E}(X_{t-1} \,|\, \mathscr{Y}_{t-1}) + \mathbf{b}\, \mathsf{E}(W_{t-1}) \\ &= \mathbf{a} \hat{x}_{t-1} \end{aligned}$$

The covariance matrix is

$$\begin{aligned} \mathsf{var}(X_t \,|\, \mathscr{Y}_{t-1}) &= \mathsf{var}(\mathbf{a} X_{t-1} + \mathbf{b} W_{t-1} \,|\, \mathscr{Y}_{t-1}) \\ &= \mathsf{var}(\mathbf{a} X_{t-1} \,|\, \mathscr{Y}_{t-1}) + \mathsf{var}(\mathbf{b} W_{t-1}) \\ &= \mathbf{a}\, \mathbf{p}_{t-1}\, \mathbf{a}' + \mathbf{b}\, \mathbf{q}\, \mathbf{b}' \end{aligned}$$

Putting these results together, we have the distribution

$$X_t \,|\, \mathscr{Y}_{t-1} \sim \text{Normal}(\underbrace{\mathbf{a} \hat{x}_{t-1}}_{=: \hat{x}_t^-}, \underbrace{\mathbf{a}\, \mathbf{p}_{t-1}\, \mathbf{a}' + \mathbf{b}\, \mathbf{q}\, \mathbf{b}'}_{=: \mathbf{p}_t^-})$$

5.3.2 Update

It remains now to update $X_t \,|\, \mathscr{Y}_{t-1}$ using a measurement y_t that is a realisation of

$$Y_t = \mathbf{c} X_t + V_t$$

This is an estimation problem like the one studied in §1.6. Indeed, $X_t \,|\, \mathscr{Y}_{t-1}$ is independent of $V_t \,|\, \mathscr{Y}_{t-1}$ (which is V_t), with $X_t \,|\, \mathscr{Y}_{t-1} \sim \text{Normal}(\hat{x}_t^-,\, \mathbf{p}_t^-)$ and $V_t \sim \text{Normal}(0,\, \mathbf{r})$. Then, as in §1.6, the posterior distribution is

$$X_t \,|\, \mathscr{Y}_t \sim \text{Normal}(\hat{x}_t,\, \mathbf{p}_t) \tag{13}$$

with

$$\begin{aligned} \mathbf{k}_t &= \mathbf{p}_t^- \mathbf{c}' \left(\mathbf{c}\, \mathbf{p}_t^- \mathbf{c}' + \mathbf{r} \right)^{-1} \\ \hat{x}_t &= \hat{x}_t^- + \mathbf{k}_t \left(y_t - \mathbf{c} \hat{x}_t^- \right) \\ \mathbf{p}_t &= \mathbf{p}_t^- - \mathbf{k}_t\, \mathbf{c}\, \mathbf{p}_t^- \end{aligned}$$

A single prediction+update step can be coded in Matlab/Octave as

```
function [x,p] = kalmanstep(x,p,y,a,b,c,q,r)
  x=a*x;
  p=a*p*a'+b*q*b';
  s=c*p*c'+r;
  k=p*c'/s;
  x=x+k*(y-c*x);
  p=p-k*s*k';
```

5.3.3 Error

The posterior mean $\hat{x}_t$ in (13) is "optimal" in the sense that no other value $z \in \mathbb{R}^{n_x}$ gives a smaller mean square error $\|X_t \,|\, \mathscr{Y}_t - z\|^2_{\text{rms}}$.

Note that the Kalman filter formulas define a determistic mapping $y_{1:t} \mapsto \hat{x}_t$ that takes real-valued observations and produces real-valued vectors. The transformation of the rv $Y_{1:t}$ by this deterministic mapping is a rv which is conventionally denoted $\mathsf{E}(X_t \,|\, Y_{1:t})$; for brevity we shall denote it $\hat{X}_t$. The (unconditional) filter error is defined as the difference

$$E_t = X_t - \hat{X}_t$$

The conditional filter error $E_t \,|\, \mathscr{Y}_t$ has cdf

$$f_{E_t \,|\, Y_{1:t}}(e \,|\, y_{1:t}) = \mathsf{P}(X_t - \hat{X}_t \le e \,|\, \mathscr{Y}_t) = \mathsf{P}(X_t \le \hat{x}_t + e \,|\, \mathscr{Y}_t) = f_{\text{Normal}(0,\mathbf{p}_t)}(e)$$

that is, $E_t \,|\, \mathscr{Y}_t \sim \text{Normal}(0, \mathbf{p}_t)$. For this reason, the matrix $\mathbf{p}_t$ is called the *error covariance*.

Because the computation of $\mathbf{p}_t$ does not involve any values of the realised measurements, the pdf of the unconditional error is given by

$$\begin{aligned} p_{E_t}(e) &= \int p_{E_t \,|\, Y_{1:t}}(e \,|\, y_{1:t}) p_{Y_{1:t}}(y_{1:t})\, \mathrm{d}y_{1:t} \\ &= p_{\text{Normal}(0,\mathbf{p}_t)}(e) \underbrace{\int p_{Y_{1:t}}(y_{1:t})\, \mathrm{d}y_{1:t}}_{=1} \end{aligned}$$

that is, the unconditional filter error E_t has the same probability distribution as the conditional filter error $E_t \,|\, \mathscr{Y}_t$.

5.3.4 Example: Tracking a Moving Target

Let the two components of X denote the position and velocity of a moving target. Assume that velocity is a standard normal random walk. The motion model is then

$$X_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} W_t, \quad W_t \sim \text{Normal}(0,\ 1)$$

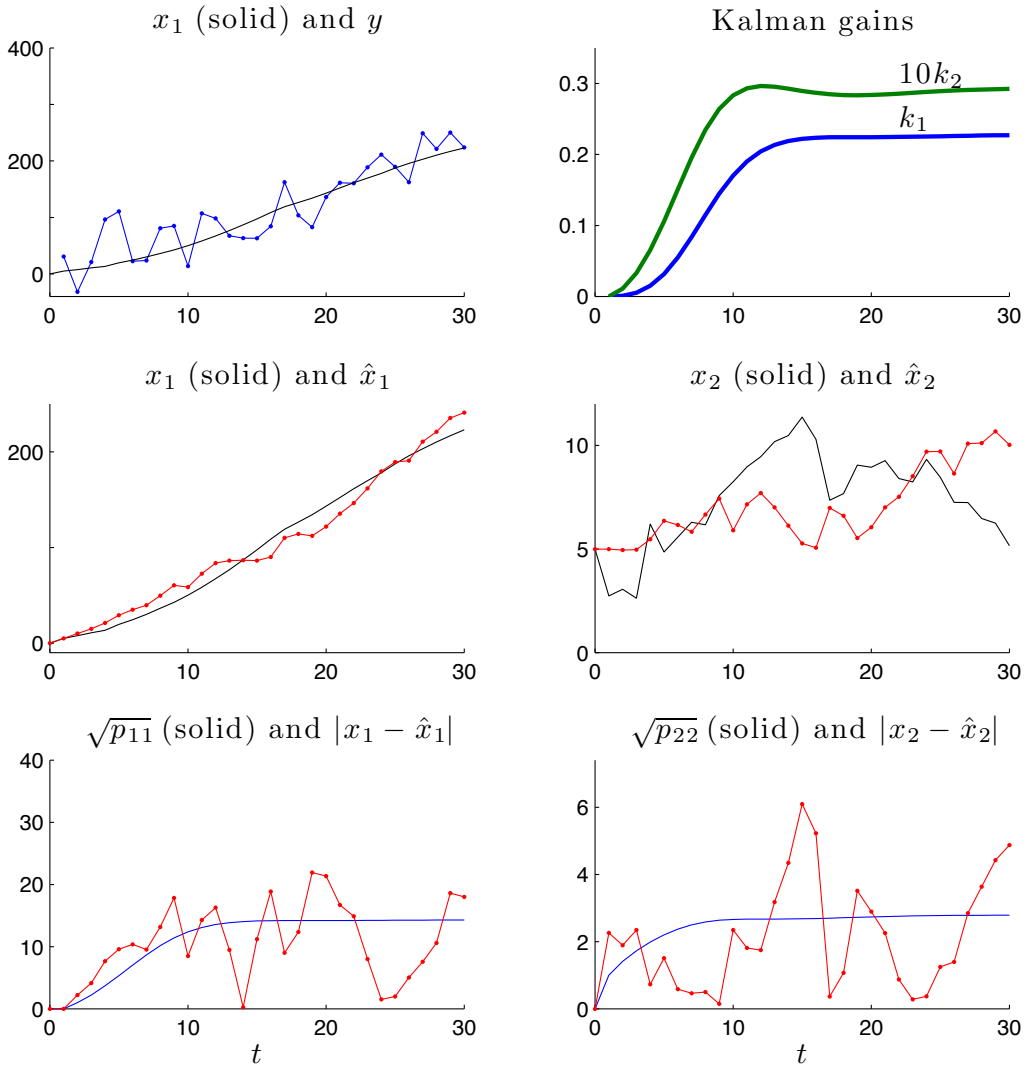
We want to estimate the motion's position and velocity from noisy measurements of the position. The measurement model is

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} X_t + V_t, \quad V_t \sim \text{Normal}(0, 900)$$

The initial state is $X_0 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, assumed to be known exactly (i.e. $\mathbf{p}_0 = \mathbf{0}$).

The results obtained using a computer-generated sample path are shown in the following figure. Notice how the Kalman gains and state variances appear to converge to steady state values, even though the motion model is not stable.

Bear in mind when looking at the plot of posterior means that each estimate value is computed using current and past measurements — the recursive filter doesn't use future measurements. (Recursive algorithms that compute mean and covariance of the state conditional on past, present and some future measurements are called *smoothers*.)

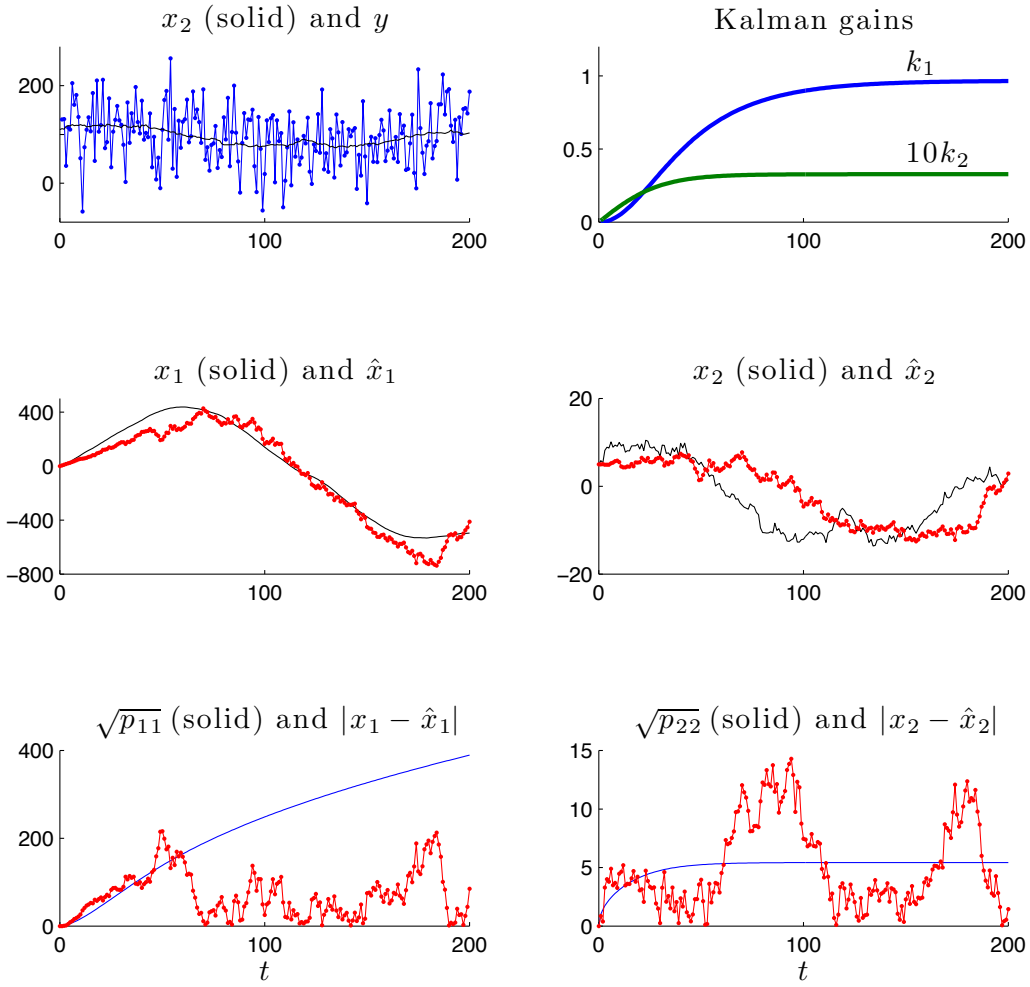


5.4 Stability and Steady-State Filter

In the Kalman filter formulas, the covariance matrices $\mathbf{p}_t^-$ and $\mathbf{p}_t$ as well as the gain matrix $\mathbf{k}_t$ evolve in time following recursive formulas that do not depend on the observation realisations y_t. The filter is said to be *stable* if these matrix sequences are convergent. In this section we present (without proofs) conditions for filter stability, show how to compute the sequence limits, and illustrate how these are used in the *steady-state Kalman filter*.

5.4.1 Stability

Consider a modified tracking system where we observe velocities instead of positions, that is, $\mathbf{c} = [0\ 1]$. The filter for this modified tracking system is unstable, because the position error covariance $\mathbf{p}_{11}$ grows without bound:



Here is some general theory (without proofs). By redefining $\mathbf{b}$ if necessary, we can assume that $\mathbf{q} = \mathbf{i}$ in the filter state space model. Then a sufficient condition for filter stability is that the pair $(\mathbf{a}, \mathbf{b})$ is *stabilizable* (that is, $\rho(\mathbf{a} - \mathbf{k}_s\mathbf{b}) < 1$ for some $\mathbf{k}_s$) and that the pair $(\mathbf{a}, \mathbf{c})$ is *detectable* (that is, $\rho(\mathbf{a} - \mathbf{c}\mathbf{k}_d) < 1$ for some $\mathbf{k}_d$). A sufficient condition for $(\mathbf{a}, \mathbf{b})$ to be stabilizable is that it is *reachable*, that is,

$$\operatorname{rank}[\,\mathbf{b} \quad \mathbf{ab} \quad \cdots \quad \mathbf{a}^{n_x-1}\mathbf{b}\,] = n_x$$

A sufficient condition for $(\mathbf{a}, \mathbf{c})$ to be detectable is that it is *reconstructible*, that is,

$$\operatorname{rank}\begin{bmatrix} \mathbf{c} \\ \mathbf{ca} \\ \vdots \\ \mathbf{ca}^{n_x-1} \end{bmatrix} = n_x$$

In the tracking example (§5.3.4), the filter is stable because

$$\text{rank}[\mathbf{b}\ \mathbf{ab}] = \text{rank}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = 2, \qquad \text{rank}\begin{bmatrix} \mathbf{c} \\ \mathbf{ca} \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 2$$

For the modified tracking problem presented at the beginning of this section, $(\mathbf{a}, \mathbf{c})$ is not reconstructible, because

$$\text{rank}\begin{bmatrix} \mathbf{c} \\ \mathbf{ca} \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

5.4.2 Steady State

The steady-state solution satisfies

$$\begin{aligned} \mathbf{p}_\infty^- &= \mathbf{a}\,\mathbf{p}_\infty\,\mathbf{a}' + \mathbf{b}\,\mathbf{q}\,\mathbf{b}' \\ \mathbf{k}_\infty &= \mathbf{p}_\infty^-\,\mathbf{c}'\left(\mathbf{c}\,\mathbf{p}_\infty^-\,\mathbf{c}' + \mathbf{r}\right)^{-1} \\ \mathbf{p}_\infty &= \mathbf{p}_\infty^- - \mathbf{k}_\infty\,\mathbf{c}\,\mathbf{p}_\infty^- \end{aligned}$$

Eliminating $\mathbf{k}_\infty$ and $\mathbf{p}_\infty$, and denoting $\mathbf{x} := \mathbf{p}_\infty^-$, we obtain the matrix equation

$$\mathbf{x} = \mathbf{a}(\mathbf{x} - \mathbf{x}\mathbf{c}'(\mathbf{c}\mathbf{x}\mathbf{c}' + \mathbf{r})^{-1}\mathbf{c}\mathbf{x})\mathbf{a}' + \mathbf{b}\mathbf{q}\mathbf{b}'$$

This matrix equation is known as a *discrete-time algebraic Riccati equation*. It can be solved using the Matlab Control Toolbox function `dare`. For the tracking example the commands

```
a=[1 1; 0 1]; c=[1 0]; b=[0;1]; r=900;
Pprior=dare(a',c',b*b',r)
k=Pprior*c'/(c*Pprior*c'+r)
P=Pprior-k*c*Pprior
```

give

$$\mathbf{p}_\infty^- = \begin{bmatrix} 265.55 & 34.14 \\ 34.14 & 8.78 \end{bmatrix}, \quad \mathbf{k}_\infty = \begin{bmatrix} 0.2278 \\ 0.0293 \end{bmatrix}, \quad \mathbf{p}_\infty = \begin{bmatrix} 205.05 & 26.36 \\ 26.36 & 7.78 \end{bmatrix}$$

For the modified tracking system in which velocities are observed instead of positions, the commands

```
a=[1 1; 0 1]; c=[0 1]; b=[0;1]; r=900;
Pprior=dare(a',c',b*b',r)
```

return an error message

```
??? Error using ==> dare at 102
Unable to solve the specified Riccati equation because
the Symplectic spectrum is too close to the unit circle.
```

which indicates that the Kalman filter is not stable.

5.4.3 Steady-State Kalman Filter

Because the covariance and gain matrix sequences $\mathbf{p}_t^-$, $\mathbf{p}_t$ and $\mathbf{k}_t$ do not depend on the realised measurements y_t, the matrices for time steps $1{:}t_{\mathrm{ss}}$ could be computed before-hand, with t_{ss} chosen large enough that the matrices get sufficiently close to their steady-state values (assuming of course that the filter is stable). This provides a faster implementation if reading the matrices from memory is faster than their computation.

Often, practicioners don't bother with the start-up portion of the covariance and gain sequences, and instead implement a "steady-state Kalman filter" that uses only the steady-state gain $\mathbf{k}_\infty$:

$$\bar{x}_0 = \hat{\mathbf{x}}_0, \quad \bar{x}_t = \mathbf{a}\bar{x}_{t-1} + \mathbf{k}_\infty(y_t - \mathbf{ca}\bar{x}_{t-1}) \quad (t \in \mathbb{Z}_+)$$

This is not optimal (unless $\mathbf{p}_0 = \mathbf{p}_\infty$), but it is simpler and easier to implement. The non-optimality is mainly evident during the Kalman filter's start-up time, after which the performance of the steady-state filter is practically the same as that of the Kalman filter. This is seen in the results for the target tracking example: the actual estimation errors are visibly larger in the filter start-up period $t \lesssim 14$, after which the errors are similar to those of the Kalman filter.